

# JORDAN GROUPS AND AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES

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**ABSTRACT.** The first section of this paper is focused on Jordan groups in abstract setting, the second on that in the settings of automorphisms groups and groups of birational self-maps of algebraic varieties. The appendix contains formulations of some open problems and the relevant comments.

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*Key words:* Jordan, Cremona, automorphism, birational map

This is the expanded version of my talk, based on [Po10, Sect. 2], at the workshop *Groups of Automorphisms in Birational and Affine Geometry*, October 29–November 3, 2012, Levico Terme, Italy. The appendix is the expanded version of my notes on open problems posted on the site of this workshop [Po12<sub>2</sub>].

Below  $k$  is an algebraically closed field of characteristic zero. Variety means algebraic variety over  $k$  in the sense of Serre (so algebraic group means algebraic group over  $k$ ). We use without explanation standard notation and conventions of [Bo91] and [Sp98]. In particular,  $k(X)$  denotes the field of rational functions of an irreducible variety  $X$ .  $\text{Bir}(X)$  denotes the group of birational self-maps of an irreducible variety  $X$ . Recall that if  $X$  is the affine  $n$ -dimensional space  $\mathbf{A}^n$ , then  $\text{Bir}(X)$  is called the *Cremona group over  $k$  of rank  $n$* ; we denote it by  $\text{Cr}_n$  (cf. [Po11], [Po12<sub>1</sub>]).

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## 1. JORDAN GROUPS

**1.1. Main definition.** The notion of Jordan group was introduced in [Po10]:

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**Definition 1** ([Po10, Def. 2.1]). A group  $G$  is called a *Jordan group* if there exists a positive integer  $d$ , depending on  $G$  only, such that every finite subgroup  $K$  of  $G$  contains a normal abelian subgroup whose index in  $K$  is at most  $d$ . The minimal such  $d$  is called the *Jordan constant of  $G$*  and is denoted by  $J_G$ .

Informally, this means that all finite subgroups of  $G$  are “almost” abelian in the sense that they are extensions of abelian groups by finite groups taken from a finite list.

Actually, one obtains the same class of groups if the assumption of normality in Definition 1 is dropped. Indeed, for any group  $P$  containing a subgroup  $Q$  of finite index, there is a normal subgroup  $N$  of  $P$  such that  $[P : N] \leq [P : Q]!$  and  $N \subseteq Q$  (see, e.g., [La65, Exer. 12 to Chap. I]).

## 1.2. Examples.

1.2.1. *Jordan’s Theorem.* The first example that led to Definition 1 justifies the coined name. It is given by the classical Jordan’s theorem [Jo78] (see, e.g., [CR62, §36] for a modern exposition). In terms of Definition 1 the latter can be reformulated as follows:

**Theorem 1** (C. Jordan, 1878). *The group  $\mathbf{GL}_n(k)$  is Jordan for every  $n$ .*

Since the symmetric group  $\mathrm{Sym}_{n+1}$  admits a faithful  $n$ -dimensional representation and the alternating group  $\mathrm{Alt}_{n+1}$  is the only non-identity proper normal subgroup of  $\mathrm{Sym}_{n+1}$  for  $n \geq 2$ ,  $n \neq 3$ , Definition 1 yields the lower bound

$$(n + 1)! \leq J_{\mathbf{GL}_n(k)} \quad \text{for } n \geq 4. \quad (1)$$

Frobenius, Schur, and Blichfeldt initiated exploration of the upper bounds for  $J_{\mathbf{GL}_n(k)}$ . In 2007, using the classification of finite simple groups, M. J. Collins [Co07] gave optimal upper bounds and thereby found the precise values of  $J_{\mathbf{GL}_n(k)}$  for all  $n$ . In particular, in [Co07] is proved that

- (i) the equality in (1) holds for all  $n \geq 71$  and  $n = 63, 65, 67, 69$ ;
- (ii)  $J_{\mathbf{GL}_n(k)} = 60^r r!$  if  $n = 2r$  or  $2r + 1$  and either  $20 \leq n \leq 62$  or  $n = 64, 66, 68, 70$ ;
- (iii)  $J_{\mathbf{GL}_n(k)} = 60, 360, 25920, 25920, 6531840$  resp., for  $n = 2, 3, 4, 5, 6$ .

The values of  $J_{\mathbf{GL}_n(k)}$  for  $7 \leq n \leq 19$  see in [Co07].

1.2.2. *Affine algebraic groups.* Since any subgroup of a Jordan groups is Jordan, Theorem 1 yields

**Corollary 1.** *Every linear group is Jordan.*

Since every affine algebraic group is linear [Sp98, 2.3.7], this, in turn, yields the following generalization of Theorem 1:

**Theorem 2.** *Every affine algebraic group is Jordan.*

1.2.3. *Nonlinear Jordan groups.* Are there nonlinear Jordan groups? The next example, together with Theorem 1, convinced me that Definition 1 singles out an interesting class of groups and therefore deserves to be introduced.

**Example 1.** By [Se09<sub>1</sub>, Thm. 5.3], [Se08, Thm. 3.1], the planar Cremona group  $\text{Cr}_2$  is Jordan. On the other hand, by [CD09, Prop. 5.1] (see also [Co07, Prop. 2.2]),  $\text{Cr}_2$  is not linear. Note that in [Se09<sub>1</sub>, Thm. 5.3] one also finds a “multiplicative” upper bound for  $J_{\text{Cr}_2}$ : as is specified there, a crude computation shows that every finite subgroup  $G$  of  $\text{Cr}_2$  contains a normal abelian subgroup  $A$  of rank  $\leq 2$  with  $[G : A]$  dividing  $2^{10} \cdot 3^4 \cdot 5^2 \cdot 7$  (it is also mentioned that the exponents of 2 and 3 can be somewhat lowered, but those of 5 and 7 cannot).  $\square$

**Example 2.** Let  $F_d$  be a free group with  $d$  free generators and let  $F_d^n$  be its normal subgroup generated by the  $n$ th powers of all elements. As is known (see, e.g., [Ad11, Thm. 2]), the group  $B(d, n) := F_d/F_d^n$  is infinite for  $d \geq 2$  and odd  $n \geq 665$  (recently S. Adian announced in [Ad13] that 665 may be replaced by 100). On the other hand, by I. Schur, finitely generated linear torsion groups are finite (see, e.g., [CR62, Thm. 36.2]). Hence infinite  $B(d, n)$  is nonlinear. On the other hand, for  $d \geq 2$  and odd  $n \geq 665$ , every finite subgroup in  $B(d, n)$  is cyclic (see [Ad11, Thm. 8]); hence  $B(d, n)$  is Jordan and  $J_{B(d, n)} = 1$ .  $\square$

**Example 3.** Let  $p$  be a positive prime integer and let  $T(p)$  be a Tarski monster group, i.e., an infinite group, such that every its proper subgroup is a cyclic group of order  $p$ . By [Ol83], for big  $p$  (e.g.,  $\geq 10^{75}$ ), such a group exists.  $T(p)$  is necessarily simple and finitely generated (and, in fact, generated by every two non-commuting elements). By the same reason as in Example 2,  $T(p)$  is not linear. The definitions imply that  $T_p$  is Jordan and  $J_{T(p)} = 1$ . (I thank A. Yu. Ol’shanskii who drew in [Ol13] my attention to this example.)  $\square$

1.2.4. *Diffeomorphism groups of smooth topological manifolds.* Let  $M$  be a compact connected  $n$ -dimensional smooth topological manifold. Assume that  $M$  admits an unramified covering  $\widetilde{M} \rightarrow M$  such that  $H^1(\widetilde{M}, \mathbf{Z})$  contains the cohomology classes  $\alpha_1, \dots, \alpha_n$  satisfying  $\alpha_1 \cup \dots \cup \alpha_n \neq 0$ . Then, by [MiR10, Thm. 1.4(1)], the group  $\text{Diff}(M)$  is Jordan. This result is applicable to  $\mathbf{T}^n$ , the product of  $n$  circles, and, more generally, to the connected sum  $N \# \mathbf{T}^n$ , where  $N$  is any compact connected orientable smooth topological manifold. (I thank I. Mundet i Riera who drew in [MiR13] my attention to [MiR10], [Fi11] and [Pu07].)

1.2.5. *Non-Jordan groups.* Are there non-Jordan groups?

**Example 4.** The group  $\text{Sym}_\infty$  of all permutations of  $\mathbf{Z}$  contains the alternating group  $\text{Alt}_n$  for every  $n$ . Hence  $\text{Sym}_\infty$  is non-Jordan because  $\text{Alt}_n$  is simple for  $n \geq 5$  and  $|\text{Alt}_n| = n!/2 \xrightarrow{n \rightarrow \infty} \infty$ .  $\square$

Using Example 4 one obtains a finitely generated non-Jordan group:

**Example 5.** Let  $\mathcal{N}$  be the subgroup of  $\text{Sym}_\infty$  generated by the transposition  $\sigma := (1, 2)$  and the “translation”  $\delta$  defined by the condition

$$\delta(i) = i + 1 \quad \text{for every } i \in \mathbf{Z}.$$

Then  $\delta^m \sigma \delta^{-m}$  is the transposition  $(m + 1, m + 2)$  for every  $m$ . Since the set of transpositions  $(1, 2), (2, 3), \dots, (n - 1, n)$  generates the symmetric group  $\text{Sym}_n$ , this shows that  $\mathcal{N}$  contains  $\text{Alt}_n$  for every  $n$ ; whence  $\mathcal{N}$  is non-Jordan.  $\square$

One can show that  $\mathcal{N}$  is not finitely presented. Here is an example of a finitely presented non-Jordan group which is also simple.

**Example 6.** Consider Richard J. Thompson’s group  $V$ , see [CFP96, §6]. It is finitely presented, simple and contains a subgroup isomorphic with  $\text{Sym}_n$  for every  $n \geq 2$ . The latter implies, as in Example 4, that  $V$  is non-Jordan. (I thank Vic. Kulikov who drew my attention to this example.)  $\square$

### 1.3. General properties.

1.3.1. *Subgroups, quotient groups, and products.* Exploring whether a group is Jordan or not leads to the questions on the connections between Jordaness of a group, its subgroup, and its quotient group.

**Theorem 3** ([Po10, Lemmas 2.6, 2.7, 2.8]).

- (1) *Let  $H$  be a subgroup of a group  $G$ .*
  - (i) *If  $G$  is Jordan, then  $H$  is Jordan and  $J_H \leq J_G$ .*
  - (ii) *If  $G$  is Jordan and  $H$  is normal in  $G$ , then  $G/H$  is Jordan and  $J_{G/H} \leq J_G$  in either of the cases:*
    - (a)  *$H$  is finite;*
    - (b) *the extension  $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$  splits.*
  - (iii) *If  $H$  is torsion-free, normal in  $G$ , and  $G/H$  is Jordan, then  $G$  is Jordan and  $J_G \leq J_{G/H}$ .*
- (2) *Let  $G_1$  and  $G_2$  be two groups. Then  $G_1 \times G_2$  is Jordan if and only if  $G_1$  and  $G_2$  are. In this case,  $J_{G_i} \leq J_{G_1 \times G_2} \leq J_{G_1} J_{G_2}$  for every  $i$ .*

*Proof.* (1)(i). This follows from Definition 1.

If  $H$  is normal in  $G$ , let  $\pi: G \rightarrow G/H$  be the natural projection.

(1)(ii)(a). Let  $F$  be a finite subgroup of  $G/H$ . Since  $H$  is finite,  $\pi^{-1}(F)$  is finite. Since  $G$  is Jordan,  $\pi^{-1}(F)$  contains a normal abelian subgroup  $A$  whose index is at most  $J_G$ . Hence  $\pi(A)$  is a normal abelian subgroup of  $F$  whose index in  $F$  is at most  $J_G$ .

(1)(ii)(b). By the condition, there is a subgroup  $S$  in  $G$  such that  $\pi|_S: S \rightarrow G/H$  is an isomorphism; whence the claim by (1)(i).

(1)(iii). Let  $F$  be a finite subgroup of  $G$ . Since  $H$  is torsion free,  $F \cap H = \{1\}$ ; whence  $\pi|_F: S \rightarrow \pi(F)$  is an isomorphism. Therefore, as  $G/H$  is Jordan,  $F$  contains a normal abelian subgroup whose index in  $F$  is at most  $J_{G/H}$ .

(2) If  $G := G_1 \times G_2$  is Jordan, then (1)(i) implies that  $G_1$  and  $G_2$  are Jordan and  $J_{G_i} \leq J_G$  for every  $i$ . Conversely, let  $G_1$  and  $G_2$  be Jordan. Let

$\pi_i: G \rightarrow G_i$  be the natural projection. Take a finite subgroup  $F$  of  $G$ . Then  $F_i := \pi_i(F)$  contains an abelian normal subgroup  $A_i$  such that

$$[F_i : A_i] \leq J_{G_i}. \quad (2)$$

The subgroup  $\tilde{A}_i := \pi_i^{-1}(A_i) \cap F$  is normal in  $F$  and  $F/\tilde{A}_i$  is isomorphic to  $F_i/A_i$ . From (2) we then conclude that

$$[F : \tilde{A}_i] \leq J_{G_i}. \quad (3)$$

Since  $A := \tilde{A}_1 \cap \tilde{A}_2$  is the kernel of the diagonal homomorphism

$$F \longrightarrow F/\tilde{A}_1 \times F/\tilde{A}_2$$

determined by the canonical projection  $F \rightarrow F/\tilde{A}_i$ , we infer from (3) that

$$[F : A] = |F/A| \leq |F/\tilde{A}_1 \times F/\tilde{A}_2| = |F_1/A_1||F_2/A_2| \leq J_{G_1}J_{G_2}. \quad (4)$$

By construction,  $A \subseteq A_1 \times A_2$  and  $A_i$  is abelian. Hence  $A$  is abelian as well. Since  $A$  is normal in  $F$ , the claim then follows from (4).  $\square$   $\square$

**Theorem 4.** *Let  $H$  be a normal subgroup of a group  $G$ . If  $H$  and  $G/H$  are Jordan, then any set of pairwise nonisomorphic simple nonabelian finite subgroups of  $G$  is finite.*

*Proof.* Since up to isomorphism there are only finitely many finite groups of a fixed order, Definition 1 implies that any set of pairwise nonisomorphic simple nonabelian finite subgroups of a given Jordan group is finite. This implies the claim because simplicity of a finite subgroup  $S$  of  $G$  yields that either  $S \subseteq H$  or the canonical projection  $G \rightarrow G/H$  embeds  $S$  in  $G/H$ .  $\square$   $\square$

1.3.2. *Counterexample.* For a normal subgroup  $H$  of  $G$ , it is not true, in general, that  $G$  is Jordan if  $H$  and  $G/H$  are.

**Example 7.** For every integer  $n > 0$  fix a finite group  $G_n$  with the properties:

- (i)  $G_n$  has an abelian normal subgroup  $H_n$  such that  $G_n/H_n$  is abelian;
- (ii) there is a subgroup  $Q_n$  of  $G_n$  such that the index in  $Q_n$  of every abelian subgroup of  $Q_n$  is greater or equal than  $n$ .

Such a  $G_n$  exists, see below. Now take  $G := \prod_n G_n$  and  $H := \prod_n H_n$ . Then  $H$  and  $G/H$  are abelian by (i), hence Jordan, but  $G$  is not Jordan by (ii).

The following construction from [Za10, Sect. 3] proves the existence of such a  $G_n$ . Let  $K$  be a finite commutative group of order  $n$  written additively and let  $\hat{K} := \text{Hom}(K, k^*)$  be the group of characters of  $K$  written multiplicatively. The formula

$$(\alpha, g, \ell)(\alpha', g', \ell') := (\alpha\alpha'\ell'(g), g + g', \ell\ell') \quad (5)$$

endows the set  $k^* \times K \times \hat{K}$  with the group structure. Denote by  $G_K$  the obtained group. It is embedded in the exact sequence of groups

$$\{1\} \rightarrow k^* \xrightarrow{\iota} G_K \xrightarrow{\pi} K \times \hat{K} \rightarrow \{(0, 1)\},$$

where  $\iota(\alpha) := (\alpha, 0, 1)$  and  $\pi((\alpha, g, \ell)) := (g, \ell)$ .

Thus, if one takes  $G_n := G_K$  and  $H_n := \iota(k^*)$ , then property (i) holds. Let  $\mu_n$  be the subgroup of all  $n$ th roots of unity in  $k^*$ . From (5) and  $|K| = n$  we infer that the subset  $Q_K := \mu_n \times K \times \widehat{K}$  is a subgroup of  $G_K$ . In [Za10, Sect. 3] is proved that for  $Q_n = Q_K$  property (ii) holds.  $\square$

1.3.3. *Bounded groups.* However, under certain conditions,  $G$  is Jordan if and only if  $H$  and  $G/H$  are. An example of such a condition is given in Theorem 5 below; it is based on Definition 2 below introduced in [Po10].

Given a group  $G$ , put

$$b_G := \sup_F |F|,$$

where  $F$  runs over all finite subgroups of  $G$ .

**Definition 2** ([Po10, Def. 2.9]). A group  $G$  is called *bounded* if  $b_G \neq \infty$ .

**Example 8.** Finite groups and torsion free groups are bounded.  $\square$

**Example 9.** It is immediate from Definition 2 that every extension of a bounded group by bounded is bounded.  $\square$

**Example 10.** By the classical Minkowski's theorem  $\mathbf{GL}_n(\mathbf{Z})$  is bounded (see, e.g., [Hu98, Thm. 39.4]). Since every finite subgroup of  $\mathbf{GL}_n(\mathbf{Q})$  is conjugate to a subgroup of  $\mathbf{GL}_n(\mathbf{Z})$  (see, e.g., [CR62, Thm. 73.5]), this implies that  $\mathbf{GL}_n(\mathbf{Q})$  is bounded and  $b_{\mathbf{GL}_n(\mathbf{Q})} = b_{\mathbf{GL}_n(\mathbf{Z})}$ . H. Minkowski and I. Schur obtained the following upper bound for  $b_{\mathbf{GL}_n(\mathbf{Z})}$ , see, e.g., [Hu98, §39]. Let  $\mathcal{P}(n)$  be the set of all primes  $p \in \mathbf{N}$  such that  $[n/(p-1)] > 0$ . Then

$$b_{\mathbf{GL}_n(\mathbf{Z})} \leq \prod_{p \in \mathcal{P}(n)} p^{d_p}, \quad \text{where } d_p = \sum_{i=0}^{\infty} \left[ \frac{n}{p^i(p-1)} \right]. \quad (6)$$

In particular, the right-hand side of the inequality in (6) is

$$2, 24, 48, 5760, 11520, 2903040 \quad \text{resp., for } n = 1, 2, 3, 4, 5, 6. \quad \square$$

**Example 11.** Maintain the notation and assumption of Subsection 1.2.4. If  $\chi(M) \neq 0$ , then by [MiR10, Thm. 1.4(2)], the group  $\text{Diff}(M)$  is bounded. Further information on smooth manifolds with bounded diffeomorphism groups is contained in [Pu07].  $\square$

**Example 12.** Every bounded group  $G$  is Jordan with  $J_G \leq b_G$ , and there are non-bounded Jordan groups (e.g.,  $\mathbf{GL}_n(k)$ ).  $\square$

**Theorem 5** ([Po10, Lemma 2.11]). *Let  $H$  be a normal subgroup of a group  $G$  such that  $G/H$  is bounded. Then  $G$  is Jordan if and only if  $H$  is Jordan, and in this case*

$$J_G \leq b_{G/H} J_H^{b_{G/H}}.$$

*Proof.* A proof is needed only for the sufficiency. So let  $H$  be Jordan and let  $F$  be a finite subgroup of  $G$ . By Definition 1

$$L := F \cap H \quad (7)$$

contains a normal abelian subgroup  $A$  such that

$$[L : A] \leq J_H. \quad (8)$$

Let  $g$  be an element of  $F$ . Since  $L$  is a normal subgroup of  $F$ , we infer that  $gAg^{-1}$  is a normal abelian subgroup of  $L$  and

$$[L : A] = [L : gAg^{-1}]. \quad (9)$$

The abelian subgroup

$$M := \bigcap_{g \in F} gAg^{-1}. \quad (10)$$

is normal in  $F$ . We intend to prove that  $[F : M]$  is upper bounded by a constant not depending on  $F$ . To this end, fix the representatives  $g_1, \dots, g_{|F/L|}$  of all cosets of  $L$  in  $F$ . Then (10) and normality of  $A$  in  $L$  imply that

$$M = \bigcap_{i=1}^{|F/L|} g_i A g_i^{-1}. \quad (11)$$

From (11) we deduce that  $M$  is the kernel of the diagonal homomorphism

$$L \longrightarrow \prod_{i=1}^{|F/L|} L/g_i A g_i^{-1}$$

determined by the canonical projections  $L \rightarrow L/g_i A g_i^{-1}$ . This, (9), and (8) yield

$$[L : M] \leq [L : A]^{|F/L|} \leq J_H^{|F/L|}. \quad (12)$$

Let  $\pi: G \rightarrow G/H$  be the canonical projection. By (7) the finite subgroup  $\pi(F)$  of  $G/H$  is isomorphic to  $F/L$ . Since  $G/H$  is bounded, this yields  $|F/L| \leq b_{G/H}$ . We then deduce from (12) and  $[F : M] = [F : L][L : M]$  that

$$[F : M] \leq b_{G/H} J_H^{b_{G/H}};$$

whence the claim.  $\square$

The following corollary should be compared with statement (1)(ii)(a) of Theorem 3:

**Corollary 2.** *Let  $H$  be a finite normal subgroup of a group  $G$  such that the center of  $H$  is trivial. If  $G/H$  is Jordan, then  $G$  is Jordan and*

$$J_G \leq |\text{Aut}(H)| J_{G/H}^{|\text{Aut}(H)|}.$$

*Proof.* Let  $\varphi: G \rightarrow \text{Aut}(H)$  be the homomorphism determined by the conjugating action of  $G$  on  $H$ . Triviality of the center of  $H$  yields  $H \cap \ker \varphi = \{1\}$ . Hence the restriction of the natural projection  $G \rightarrow G/H$  to  $\ker \varphi$  is an embedding  $\ker \varphi \hookrightarrow G/H$ . Therefore,  $\ker \varphi$  is Jordan since  $G/H$  is. But  $G/\ker \varphi$  is finite since it is isomorphic to a subgroup of  $\text{Aut}(H)$  for the finite group  $H$ . By Theorem 5 this implies the claim.  $\square$

## 2. WHEN ARE $\mathbf{Aut}(X)$ AND $\mathbf{Bir}(X)$ JORDAN?

**2.1. Problems A and B.** In [Po10, Sect. 2] were posed the following two problems:

**Problem A.** Describe algebraic varieties  $X$  for which  $\mathbf{Aut}(X)$  is Jordan.

**Problem B.** The same with  $\mathbf{Aut}(X)$  replaced by  $\mathbf{Bir}(X)$ .

Note that for rational varieties  $X$  Problem B means finding  $n$  such that the Cremona group  $\mathbf{Cr}_n$  is Jordan; in this case, it was essentially posed in [Se09<sub>1</sub>, 6.1].

Describing finite subgroups of the groups  $\mathbf{Aut}(X)$  and  $\mathbf{Bir}(X)$  for various varieties  $X$  is a classical research direction, currently flourishing. Understanding which of these groups are Jordan sheds a light on the structure of these subgroups. Varieties  $X$  with non-Jordan group  $\mathbf{Bir}(X)$  or  $\mathbf{Aut}(X)$  are, in a sense, more “symmetric” and, therefore, more remarkable than those with Jordan group. The discussion below supports the conclusion that they occur “rarely” and their finding is a challenge.

**2.2. Groups  $\mathbf{Aut}(X)$ .** In this subsection we shall consider Problem A.

**Lemma 1.** *Let  $X_1, \dots, X_n$  be all the irreducible components of a variety  $X$ . If every  $\mathbf{Aut}(X_i)$  is Jordan, then  $\mathbf{Aut}(X)$  is Jordan.*

*Proof.* Define the homomorphism  $\pi: \mathbf{Aut}(X) \rightarrow \mathbf{Sym}_n$  by  $g \cdot X_i = X_{\pi(g)}$  for  $g \in \mathbf{Aut}(X)$ . Then  $g \cdot X_i = X_i$  for every  $g \in \mathbf{Ker}(\pi)$  and  $i$ , so the homomorphism  $\pi_i: \mathbf{Ker}(\pi) \rightarrow \mathbf{Aut}(X_i)$ ,  $g \mapsto g|_{X_i}$ , arises. The definition implies that  $\pi_1 \times \dots \times \pi_n: \mathbf{Ker}(\pi) \rightarrow \prod_{i=1}^n \mathbf{Aut}(X_i)$  is an injection; whence  $\mathbf{Ker}(\pi)$  is Jordan by Theorem 3(2). Hence  $\mathbf{Aut}(X)$  is Jordan by Theorem 5.  $\square$

At this writing (October 2013), not a single variety  $X$  with non-Jordan  $\mathbf{Aut}(X)$  is known (to me).

**Question 1** ([Po10, Quest. 2.30 and 2.14]). Is there an irreducible variety  $X$  such that  $\mathbf{Aut}(X)$  is non-Jordan? Is there an irreducible affine variety  $X$  with this property?

**Remark 1.** One may consider the counterpart of the first question replacing  $X$  by a connected smooth topological manifold  $M$ , and  $\mathbf{Aut}(X)$  by  $\mathbf{Diff}(M)$ . The following yields the affirmative answer:

**Theorem 6** ([Po13]). *There is a simply connected noncompact smooth oriented 4-dimensional manifold  $M$  such that  $\mathbf{Diff}(M)$  contains an isomorphic copy of every finitely presented (in particular, of every finite) group. This copy acts on  $M$  properly discontinuously.*

Clearly,  $\mathbf{Diff}(M)$  is non-Jordan. By [Po13, Thm. 2], “noncompact” in Theorem 6 cannot be replaced by “compact”. The following question (I reformulate it using Definition 1) was posed by É. Ghys (see [Fi11, Quest. 13.1]): Is the diffeomorphism group of any compact smooth manifold Jordan? In fact, according to [MiR13], É. Ghys conjectured the affirmative answer.



On the other hand, in many cases it can be proven that  $\text{Aut}(X)$  is Jordan. Below are described several extensive classes of  $X$  with this property.

2.2.1. *Toral varieties.* First, consider the wide class of affine varieties singled out by the following

**Definition 3** ([Po10, Def. 1.13]). A variety is called *toral* if it is isomorphic to a closed subvariety of some  $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ , where  $H_i$  is the set of zeros of the  $i$ th standard coordinate function  $x_i$  on  $\mathbf{A}^n$ .

**Remark 2.**  $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$  is the group variety of the  $n$ -dimensional affine torus; whence the terminology. Warning: “toral” does not imply “affine toric” in the sense of [Fu93].

The class of toral varieties is closed with respect to taking products and closed subvarieties.

**Lemma 2** ([Po10, Lemma 1.14(a)]). *The following properties of an affine variety  $X$  are equivalent:*

- (i)  $X$  is toral;
- (ii)  $k[X]$  is generated by  $k[X]^*$ , the group of units of  $k[X]$ .

*Proof.* If  $X$  is closed in  $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ , then the restriction of functions is an epimorphism  $k[\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i] \rightarrow k[X]$ . Since  $k[\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i] = k[x_1, \dots, x_n, 1/x_1, \dots, 1/x_n]$ , this proves (i)  $\Rightarrow$  (ii).

Conversely, assume that (ii) holds and let

$$k[X] = k[f_1, \dots, f_n] \tag{13}$$

for some  $f_1, \dots, f_n \in k[X]^*$ . Since  $X$  is affine, (13) implies that  $\iota: X \rightarrow \mathbf{A}^n$ ,  $x \mapsto (f_1(x), \dots, f_n(x))$ , is a closed embedding. The standard coordinate functions on  $\mathbf{A}^n$  do not vanish on  $\iota(X)$  since every  $f_i$  does not vanish on  $X$ . Hence  $\iota(X) \subseteq \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ . This proves (ii)  $\Rightarrow$  (i).  $\square$   $\square$

**Lemma 3.** *Any quasiprojective variety  $X$  endowed with a finite automorphism group  $G$  is covered by  $G$ -stable toral open subsets.*

*Proof.* First, any point  $x \in X$  is contained in a  $G$ -stable affine open subset of  $X$ . Indeed, since the orbit  $G \cdot x$  is finite and  $X$  is quasiprojective, there is an affine open subset  $U$  of  $X$  containing  $G \cdot x$ . Hence  $V := \bigcap_{g \in G} g \cdot U$  is a  $G$ -stable open subset containing  $x$ , and, since every  $g \cdot U$  is affine,  $V$  is affine as well, see, e.g., [Sp98, Prop. 1.6.12(i)].

Thus, the problem is reduced to the case where  $X$  is affine. Assume then that  $X$  is affine, and let  $k[X] = k[h_1, \dots, h_s]$ . Replacing  $h_i$  by  $h_i + \alpha_i$  for an appropriate  $\alpha_i \in k$ , we may (and shall) assume that every  $h_i$  vanishes nowhere on the  $G \cdot x$ . Expanding the set  $\{h_1, \dots, h_s\}$  by including  $g \cdot h_i$  for every  $i$  and  $g \in G$ , we may (and shall) assume that  $\{h_1, \dots, h_s\}$  is  $G$ -stable. Then  $h := h_1 \cdots h_s \in k[X]^G$ . Hence the affine open set  $X_h := \{z \in X \mid h(z) \neq 0\}$  is  $G$ -stable and contains  $G \cdot x$ . Since  $k[X_h] = k[h_1, \dots, h_s, 1/h]$  and  $h_1, \dots, h_s, 1/h \in k[X_h]^*$ , the variety  $X_h$  is toral by Lemma 2.  $\square$   $\square$

**Remark 3.** Lemma 3 and its proof remain true for any variety  $X$  such that every  $G$ -orbit is contained in an affine open subset; whence the following

**Corollary 3.** *Every variety is covered by open toral subsets.*

For irreducible toral varieties the following was proved in [Po10, Thm. 2.16].

**Theorem 7.** *The automorphism group of every toral variety is Jordan.*

*Proof.* By Theorem 1 it suffices to prove this for irreducible toral varieties.

By [Ro57], for any irreducible variety  $X$ ,

$$\Gamma := k[X]^*/k^*$$

is a free abelian group of finite rank. Let  $X$  be toral and let  $H$  be the kernel of the natural action of  $\text{Aut}(X)$  on  $\Gamma$ . We claim that  $H$  is abelian. Indeed, for every function  $f \in k[X]^*$ , the line in  $k[X]$  spanned over  $k$  by  $f$  is  $H$ -stable. Since  $\mathbf{GL}_1$  is abelian, this yields that

$$h_1 h_2 \cdot f = h_2 h_1 \cdot f \quad \text{for any elements } h_1, h_2 \in H. \quad (14)$$

As  $X$  is toral,  $k[X]^*$  generates the  $k$ -algebra  $k[X]$  by Lemma 2. Hence (14) holds for every  $f \in k[X]$ . Since  $X$  is affine, the automorphisms of  $X$  coincide if and only if they induce the same automorphisms of  $k[X]$ . Therefore,  $H$  is abelian, as claimed.

Let  $n$  be the rank of  $\Gamma$ . Then  $\text{Aut}(\Gamma)$  is isomorphic to  $\mathbf{GL}_n(\mathbf{Z})$ . By the definition of  $H$ , the natural action of  $\text{Aut}(X)$  on  $\Gamma$  induces an embedding of  $\text{Aut}(X)/H$  into  $\text{Aut}(\Gamma)$ . Hence  $\text{Aut}(X)/H$  is isomorphic to a subgroup of  $\mathbf{GL}_n(\mathbf{Z})$  and therefore is bounded by Example 8(2). Thus,  $\text{Aut}(X)$  is an extension of a bounded group by an abelian group, hence Jordan by Theorem 5. This completes the proof.  $\square$

**Remark 4.** Maintain the notation of the proof of Theorem 7 and assume that  $X$  is irreducible. Let  $f_1, \dots, f_n$  be a basis of  $\Gamma$ . There are the homomorphisms  $\lambda_i: H \rightarrow k^*$ ,  $i = 1, \dots, n$ , such that  $h \cdot f_i = \lambda_i(h) f_i$  for every  $h \in H$  and  $i$ . Since  $k[X]^*$  generates  $k[X]$ , the diagonal map  $H \rightarrow (k^*)^n$ ,  $h \mapsto (\lambda_1(h), \dots, \lambda_n(h))$ , is injective. This and the proof of Theorem 7 show that for any irreducible toral variety  $X$  with  $\text{rk } k[X]^*/k^* = n$ , there is an exact sequence

$$\{1\} \rightarrow D \rightarrow \text{Aut}(X) \rightarrow B \rightarrow \{1\},$$

where  $D$  is a subgroup of the torus  $(k^*)^n$  and  $B$  is a subgroup of  $\mathbf{GL}_n(\mathbf{Z})$ .

Combining Theorem 7 with Corollary of Lemma 3, we get the following:

**Theorem 8.** *Any point of any variety has an open neighborhood  $U$  such that  $\text{Aut}(U)$  is Jordan.*

2.2.2. *Affine spaces.* Next, consider the fundamental objects of algebraic geometry, the affine spaces  $\mathbf{A}^n$ . The group  $\text{Aut}(\mathbf{A}^n)$  is the “*affine Cremona group of rank  $n$* ”.

Since  $\text{Aut}(\mathbf{A}^1)$  is the affine algebraic group  $\text{Aff}_1$ , it is Jordan by Theorem 2.

Since  $\text{Aut}(\mathbf{A}^2)$  is the subgroup of  $\text{Cr}_2$ , it is Jordan by Example 1. Another proof: By [Ig77] every finite subgroup of  $\text{Aut}(\mathbf{A}^2)$  is conjugate to a subgroup of  $\mathbf{GL}_2(k)$ , so the claim follows from Theorem 1.

The group  $\text{Aut}(\mathbf{A}^3)$  is Jordan being the subgroup of  $\text{Cr}_3$  that is Jordan by Corollary 13 below.

At this writing (October 2013) is unknown whether  $\text{Aut}(\mathbf{A}^n)$  is Jordan for  $n \geq 4$  or not. By Theorem 16, if the so-called BAB Conjecture (see Subsection 2.3.5 below) holds true in dimension  $n$ , then  $\text{Cr}_n$  is Jordan, hence  $\text{Aut}(\mathbf{A}^n)$  is Jordan as well.

**2.2.3. Fixed points and Jordaness.** The following method of proving Jordaness of  $\text{Aut}(X)$  was suggested in [Po10, Sect. 2] and provides extensive classes of  $X$  with Jordan  $\text{Aut}(X)$ . It is based on the use of the following fact:

**Lemma 4.** *Let  $X$  be an irreducible variety, let  $G$  be a finite subgroup of  $\text{Aut}(X)$ , and let  $x \in X$  be a fixed point of  $G$ . Then the natural action of  $G$  on  $\mathbb{T}_{x,X}$ , the tangent space of  $X$  at  $x$ , is faithful.*

*Proof.* Let  $\mathfrak{m}_{x,X}$  be the maximal ideal of  $\mathcal{O}_{x,X}$ , the local ring of  $X$  at  $x$ . Being finite,  $G$  is reductive. Since  $\text{char } k = 0$ , this implies that  $\mathfrak{m}_{x,X} = L \oplus \mathfrak{m}_{x,X}^2$  for some submodule  $L$  of the  $G$ -module  $\mathfrak{m}_{x,X}$ . Let  $K$  be the kernel of the action of  $G$  on  $L$  and let  $L^d$  be the  $k$ -linear span in  $\mathfrak{m}_{x,X}$  of the  $d$ th powers of all the elements of  $L$ . By the Nakayama's Lemma, the restriction to  $L^d$  of the natural projection  $\mathfrak{m}_{x,X} \rightarrow \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^{d+1}$  is surjective. Hence  $K$  acts trivially on  $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^{d+1}$  for every  $d$ .

Take an element  $f \in \mathfrak{m}_{x,X}$ . Since  $G$  is finite, the  $k$ -linear span  $\langle K \cdot f \rangle$  of the  $K$ -orbit of  $f$  in  $\mathfrak{m}_{x,X}$  is finite-dimensional. This and  $\bigcap_s \mathfrak{m}_{x,X}^s = \{0\}$  (see, e.g., [AM69, Cor. 10.18]) implies that  $\langle K \cdot f \rangle \cap \mathfrak{m}_{x,X}^{d+1} = \{0\}$  for some  $d$ . Since  $f - g \cdot f \in \mathfrak{m}_{x,X}^{d+1}$  for every element  $g \in K$ , we conclude that  $f = g \cdot f$ , i.e.,  $f$  is  $K$ -invariant. Thus,  $K$  acts trivially on  $\mathfrak{m}_{x,X}$ , hence on  $\mathcal{O}_{x,X}$  as well. Since  $k(X)$  is the field of fractions of  $\mathcal{O}_{x,X}$ ,  $K$  acts trivially on  $k(X)$ , and therefore, on  $X$ . But  $K$  acts on  $X$  faithfully because  $K \subseteq \text{Aut}(X)$ . This proves that  $K$  is trivial. Since  $L$  is the dual of the  $G$ -module  $\mathbb{T}_{x,X}$ , this completes the proof.  $\square$

The idea of the method is to use the fact that if a finite subgroup  $G$  of  $\text{Aut}(X)$  has a fixed point  $x \in X$ , then, by Lemma 4 and Theorem 1, there is a normal abelian subgroup of  $G$  whose index in  $G$  is at most  $J_{\mathbf{GL}_n(k)}$  for  $n = \dim \mathbb{T}_{x,X}$ .

This yields the following:

**Theorem 9.** *Let  $X$  be an irreducible variety and let  $G$  be a finite subgroup of  $\text{Aut}(X)$ . If  $G$  has a fixed point in  $X$ , then there is a normal abelian subgroup of  $G$  whose index in  $G$  is at most  $J_{\mathbf{GL}_m(k)}$ , where*

$$m = \max_x \dim \mathbb{T}_{x,X}. \quad (15)$$

**Corollary 4.** *If every finite automorphism group of an irreducible variety  $X$  has a fixed point in  $X$ , then  $\text{Aut}(X)$  is Jordan and*

$$J_{\text{Aut}(X)} \leq J_{\mathbf{GL}_m(k)},$$

where  $m$  is defined by (15).

**Corollary 5.** *Let  $p$  be a prime number. Then every finite  $p$ -subgroup  $G$  of  $\text{Aut}(\mathbf{A}^n)$  contains an abelian normal subgroup whose index in  $G$  is at most  $J_{\mathbf{GL}_n(k)}$ .*

*Proof.* This follows from Theorem 9 since in this case  $(\mathbf{A}^n)^G \neq \emptyset$ , see [Se09<sub>2</sub>, Thm. 1.2].  $\square$

**Remark 5.** At this writing (October 2013), it is unknown whether or not  $(\mathbf{A}^n)^G \neq \emptyset$  for every finite subgroup  $G$  of  $\text{Aut}(\mathbf{A}^n)$ . By Theorem 9 the affirmative answer would imply that  $\text{Aut}(\mathbf{A}^n)$  is Jordan (cf. Subsection (2.2.2)).

**Remark 6.** The statement of Corollary 5 remains true if  $\mathbf{A}^n$  is replaced by any  $p$ -acyclic variety  $X$ , and  $n$  in  $J_{\mathbf{GL}_n(k)}$  is replaced by  $m$  (see (15)). This is because in this case  $X^G \neq \emptyset$  for every finite  $p$ -subgroup  $G$  of  $\text{Aut}(X)$ , see [Se09<sub>2</sub>, Sect. 7–8].

The following applications are obtained by combining the above idea with Theorem 5.

**Theorem 10.** *Let  $X$  be an irreducible variety. Consider an  $\text{Aut}(X)$ -stable equivalence relation  $\sim$  on the set its points. If there is a finite equivalence class  $C$  of  $\sim$ , then  $\text{Aut}(X)$  is Jordan and*

$$J_{\text{Aut}(X)} \leq |C|! J_{\mathbf{GL}_m(k)}^{|C|},$$

where  $m$  is defined by (15).

*Proof.* By the assumption, every equivalence class of  $\sim$  is  $\text{Aut}(X)$ -stable. The kernel  $K$  of the action of  $\text{Aut}(X)$  on  $C$  is a normal subgroup of  $\text{Aut}(X)$  and, since the elements of  $\text{Aut}(X)$  induce permutations of  $C$ ,

$$[\text{Aut}(X) : K] \leq |C|!. \quad (16)$$

By Theorem 5, Jordaness of  $\text{Aut}(X)$  follows from that of  $K$ . To prove that the latter holds, take a point of  $x \in C$ . Since  $x$  is fixed by every finite subgroup of  $K$ , Theorem 9 implies that  $K$  is Jordan and  $J_K \leq J_{\mathbf{GL}_m(k)}$ . By Theorem 5, this and (16) imply the claim.  $\square$

**Example 13.** Below are several examples of  $\text{Aut}(X)$ -stable equivalence relations on an irreducible variety  $X$ :

- (i)  $x \sim y \iff \mathcal{O}_{x,X}$  and  $\mathcal{O}_{y,X}$  are  $k$ -isomorphic;
- (ii)  $x \sim y \iff \dim T_{x,X} = \dim T_{y,X}$ ;
- (iii)  $x \sim y \iff$  the tangent cones of  $X$  at  $x$  and  $y$  are isomorphic.  $\square$

**Corollary 6.** *If an irreducible variety  $X$  has a point  $x$  such that the set*

$$\{y \in X \mid \mathcal{O}_{x,X} \text{ and } \mathcal{O}_{y,X} \text{ are } k\text{-isomorphic}\}$$

*is finite, then  $\text{Aut}(X)$  is Jordan.*

Call a point  $x \in X$  a *vertex* of  $X$  if

$$\dim T_{x,X} \geq \dim T_{y,X} \text{ for every point } y \in X.$$

Thus every point of  $X$  is a vertex of  $X$  if and only if  $X$  is smooth.

**Corollary 7.** *The automorphism group of every irreducible variety with only finitely many vertices is Jordan.*

**Corollary 8.** *The automorphism group of every nonsmooth irreducible variety with only finitely many singular points is Jordan.*

**Corollary 9.** *Let  $X \subset \mathbf{A}^n$  be the affine cone of a smooth closed proper irreducible subvariety  $Z$  of  $\mathbf{P}^{n-1}$  that does not lie in any hyperplane. Then  $\text{Aut}(X)$  is Jordan.*

*Proof.* The assumptions imply that the singular locus of  $X$  consists of a single point, the origin; whence the claim by Corollary 8.  $\square$   $\square$

**Corollary 10.** *If an irreducible variety  $X$  has a point  $x$  such that there are only finitely many points  $y \in X$  for which the tangent cones of  $X$  at  $x$  and at  $y$  are isomorphic, then  $\text{Aut}(X)$  is Jordan.*

**Remark 7.** Smoothness in Corollary 9 may be replaced by the assumption that  $Z$  is not a cone. Indeed, in this case the origin constitutes a single equivalence class of equivalence relation (iii) in Example 13; whence the claim by Corollary 10.

2.2.4. *The Koras–Russell threefolds.* Let  $X = X_{d,s,l}$  be the so-called Koras–Russell threefold of the first kind [M-J11], i.e., the smooth hypersurface in  $\mathbf{A}^4$  defined by the equation

$$x_1^d x_2 + x_3^s + x_4^l + x_1 = 0,$$

where  $d \geq 2$  and  $2 \leq s \leq l$  with  $s$  and  $l$  relatively prime; the case  $d = s = 2$  and  $l = 3$  is the famous Koras–Russell cubic. According to [M-J11, Cor. 6.1], every element of  $\text{Aut}(X)$  fixes the origin  $(0, 0, 0, 0) \in X$ . By Corollary 4 and item (iii) of Subsection 1.2.1 this implies that  $\text{Aut}(X)$  is Jordan and

$$J_{\text{Aut}(X)} \leq 360.$$

Actually, during the conference I learned from L. Moser-Jauslin that  $X$  contains a line  $\ell$  passing through the origin, stable with respect to  $\text{Aut}(X)$ , and such that every element of  $\text{Aut}(X)$  fixing  $\ell$  pointwise has infinite order. This implies that every finite subgroup of  $\text{Aut}(X)$  is cyclic and hence

$$J_{\text{Aut}(X)} = 1.$$

2.2.5. *Small dimensions.* Since  $\text{Aut}(X)$  is a subgroup of  $\text{Bir}(X)$ , Jordaness of  $\text{Bir}(X)$  implies that of  $\text{Aut}(X)$ . This and Theorem 14 below yield the following

**Theorem 11.** *Let  $X$  be an irreducible variety of dimension  $\leq 2$  not birationally isomorphic to  $\mathbf{P}^1 \times E$ , where  $E$  is an elliptic curve. Then  $\text{Aut}(X)$  is Jordan.*

Note that if  $E$  is an elliptic curve and  $X = \mathbf{P}^1 \times E$ , then  $\text{Aut}(X) = \text{PGL}_2(k) \times \text{Aut}(E)$ , see [Ma71, pp. 98–99]. Fixing a point of  $E$ , endow  $E$  with a structure of abelian variety  $E_{\text{ab}}$ . Since  $\text{Aut}(E)$  is an extension of the finite group  $\text{Aut}(E_{\text{ab}})$  by the abelian group  $E_{\text{ab}}$ , Theorems 2, 5, and 3(2) imply that  $\text{Aut}(\mathbf{P}^1 \times E)$  is Jordan.

Note also that all irreducible curves (not necessarily smooth and projective) whose automorphism group is infinite are classified in [Po78].

2.2.6. *Non-uniruled varieties.* Again, using that Jordaness of  $\text{Bir}(X)$  implies that of  $\text{Aut}(X)$ , we deduce from recent Theorem 17(i)(a) below the following

**Theorem 12.**  *$\text{Aut}(X)$  is Jordan for any irreducible non-uniruled variety  $X$ .*

2.3. **Groups  $\text{Bir}(X)$ .** Now we shall consider Problem B (see Subsection 2.1). Exploring  $\text{Bir}(X)$ , one may, maintaining this group, replace  $X$  by any variety birationally isomorphic to  $X$ . Note that by Theorem 8 one can always attain that after such a replacement  $\text{Aut}(X)$  becomes Jordan.

The counterpart of Question 1 is

**Question 2** ([Po10, Quest. 2.31]). Is there an irreducible variety  $X$  such that  $\text{Bir}(X)$  is non-Jordan?

In contrast to the case of Question 1, at present we know the answer to Question 2: motivated by my question, Yu. Zarhin proved in [Za10] the following

**Theorem 13** ([Za10, Cor. 1.3]). *Let  $X$  be an abelian variety of positive dimension and let  $Z$  be a rational variety of positive dimension. Then  $\text{Bir}(X \times Z)$  is non-Jordan.*

*Sketch of proof.* By Theorem 3(1)(i), it suffices to prove that  $\text{Bir}(X \times \mathbf{A}^1)$  is non-Jordan. Consider an ample divisor  $D$  on  $X$  and the sheaf  $L := \mathcal{O}_X(D)$ . For a positive integer  $n$ , consider the following group  $\Theta(L^n)$ . Its elements are all pairs  $(x, [f])$  where  $x \in X$  is such that  $L^n \cong T_x^*(L^n)$  for the translation  $T_x: X \rightarrow X$ ,  $z \mapsto z + x$ , and  $[f]$  is the automorphism of the additive group of  $k(X)$  induced by the multiplication by  $f \in k(X)^*$ . The group structure of  $\Theta(L^n)$  is defined by  $(x, [f])(y, [h]) = (x + y, [T_x^*h \cdot f])$ . One proves that  $\Theta(L^n)$  enjoys the properties: (i)  $\varphi: \Theta(L^n) \rightarrow \text{Bir}(X \times \mathbf{A}^1)$ ,  $\varphi(x, [f])(y, t) = (x + y, f(y)t)$ , is a group embedding; (ii)  $\Theta(L^n)$  is isomorphic to a group  $G_K$  from Example 7 with  $|K| \geq n$ . This implies the claim (see Example 7).  $\square$

Below Problem B is solved for varieties of small dimensions ( $\leq 2$ ).

2.3.1. *Curves.* If  $X$  is a curve, then the answer to Question 2 is negative.

Proving this, we may assume that  $X$  is smooth and projective; whence  $\text{Bir}(X) = \text{Aut}(X)$ .

If  $g(X)$ , the genus of  $X$ , is 0, then  $X = \mathbf{P}^1$ , so  $\text{Aut}(X) = \mathbf{PGL}_2(k)$ . Hence  $\text{Aut}(X)$  is Jordan by Theorem 2.

If  $g(X) = 1$ , then  $X$  is an elliptic curve, hence  $\text{Aut}(X)$  is Jordan (see the penultimate paragraph in Subsection 2.2.5).

If  $g(X) \geq 2$ , then, being finite,  $\text{Aut}(X)$  is Jordan.

2.3.2. *Surfaces.* Answering Question 2 for surfaces  $X$ , we may assume that  $X$  is a smooth projective minimal model.

If  $X$  is of general type, then by Matsumura's theorem  $\text{Bir}(X)$  is finite, hence Jordan.

If  $X$  is rational, then  $\text{Bir}(X)$  is  $\text{Cr}_2$ , hence Jordan, see Example 1.

If  $X$  is a nonrational ruled surface, it is birationally isomorphic to  $\mathbf{P}^1 \times B$  where  $B$  is a smooth projective curve such that  $g(B) > 0$ ; we may then take  $X = \mathbf{P}^1 \times B$ . Since  $g(B) > 0$ , there are no dominant rational maps  $\mathbf{P}^1 \dashrightarrow B$ ; whence the elements of  $\text{Bir}(X)$  permute fibers of the natural projection  $\mathbf{P}^1 \times B \rightarrow B$ . The set of elements inducing trivial permutation is a normal subgroup  $\text{Bir}_B(X)$  of  $\text{Bir}(X)$ . The definition implies that  $\text{Bir}_B(X) = \mathbf{PGL}_2(k(B))$ , hence  $\text{Bir}_B(X)$  is Jordan by Theorem 2. Identifying  $\text{Aut}(B)$  with the subgroup of  $\text{Bir}(X)$  in the natural way, we get the decomposition

$$\text{Bir}(X) = \text{Bir}_B(X) \rtimes \text{Aut}(B). \quad (17)$$

If  $g(B) \geq 2$ , then  $\text{Aut}(B)$  is finite; whence  $\text{Bir}(X)$  is Jordan by virtue of (17) and Theorem 5. If  $g(B) = 1$ , then  $\text{Bir}(X)$  is non-Jordan by Theorem 13.

The canonical class of all the other surfaces  $X$  is numerically effective, so, for them,  $\text{Bir}(X) = \text{Aut}(X)$ , cf. [IS96, Sect. 7.1, Thm. 1 and Sect. 7.3, Thm. 2].

Let  $X$  be such a surface. The group  $\text{Aut}(X)$  has a structure of a locally algebraic group with finite or countably many components, see [Ma58], i.e., there is a normal subgroup  $\text{Aut}(X)^0$  in  $\text{Aut}(X)$  such that

- (i)  $\text{Aut}(X)^0$  is a connected algebraic group,
- (ii)  $\text{Aut}(X)/\text{Aut}(X)^0$  is either a finite or a countable group,

By (i) and the structure theorem on algebraic groups [Ba55], [Ro56] there is a normal connected affine algebraic subgroup  $L$  of  $\text{Aut}(X)^0$  such that  $\text{Aut}(X)^0/L$  is an abelian variety. By [Ma63, Cor. 1] nontriviality of  $L$  would imply that  $X$  is ruled. Since we assumed that  $X$  is not ruled, this means that  $\text{Aut}(X)^0$  is an abelian variety. Hence  $\text{Aut}(X)^0$  is abelian and, a fortiori, Jordan.

By (i) the group  $\text{Aut}(X)^0$  is contained in the kernel of the natural action of  $\text{Aut}(X)$  on  $H^2(X, \mathbf{Q})$  (we may assume that  $k = \mathbf{C}$ ). Therefore, this action defines a homomorphism  $\text{Aut}(X)/\text{Aut}(X)^0 \rightarrow \mathbf{GL}(H^2(X, \mathbf{Q}))$ . The kernel of this homomorphism is finite by [Do86, Prop. 1], and the image is bounded by Example 10. By Examples 8, 9 this yields that  $\text{Aut}(X)/\text{Aut}(X)^0$  is bounded. In turn, since  $\text{Aut}(X)^0$  is Jordan, by Theorem 5 this implies that  $\text{Aut}(X)$  is Jordan.

2.3.3. *The upshot.* The upshot of the last two subsections is

**Theorem 14** ([Po10, Thm. 2.32]). *Let  $X$  be an irreducible variety of dimension  $\leq 2$ . Then the following two properties are equivalent:*

- (a) *the group  $\text{Bir}(X)$  is Jordan;*
- (b) *the variety  $X$  is not birationally isomorphic to  $\mathbf{P}^1 \times B$ , where  $B$  is an elliptic curve.*

2.3.4. *Finite and connected algebraic subgroups of  $\text{Bir}(X)$  and  $\text{Aut}(X)$ .* Recall that the notions of algebraic subgroup of  $\text{Bir}(X)$  and  $\text{Aut}(X)$  make sense, and every algebraic subgroup of  $\text{Aut}(X)$  is that of  $\text{Bir}(X)$ , see, e.g., [Po11, Sect. 1]. Namely, a map  $\psi: S \rightarrow \text{Bir}(X)$  of a variety  $S$  is called an *algebraic family* if the domain of definition of the partially defined map  $\alpha: S \times X \rightarrow X$ ,  $(s, x) \mapsto \psi(s)(x)$  contains a dense open subset of  $S \times X$  and

$\alpha$  coincides on it with a rational map  $\varrho: S \times X \dashrightarrow X$ . The group  $\text{Bir}(X)$  is endowed with the *Zariski topology* [Se08, Sect. 1.6], in which a subset  $Z$  of  $\text{Bir}(X)$  is closed if and only if  $\psi^{-1}(Z)$  is closed in  $S$  for every family  $\psi$ . If  $S$  is an algebraic group and  $\psi$  is an algebraic family which is a homomorphism of abstract groups, then  $\psi(S)$  is called an *algebraic subgroup of  $\text{Bir}(X)$* . In this case,  $\ker(\psi)$  is closed in  $S$  and the restriction to  $\psi(S)$  of the Zariski topology of  $\text{Bir}(X)$  coincides with the topology determined by the natural identification of  $\psi(S)$  with the algebraic group  $S/\ker(\psi)$ . If  $\psi(S) \subset \text{Aut}(X)$  and  $\varrho$  is a morphism, then  $\psi(S)$  is called an *algebraic subgroup of  $\text{Aut}(X)$* .

The following reveals a relation between embeddability of finite subgroups of  $\text{Bir}(X)$  in connected affine algebraic subgroups of  $\text{Bir}(X)$  and Jordaness of  $\text{Bir}(X)$  (and the same holds for  $\text{Aut}(X)$ ).

For every integer  $n > 0$ , consider the set of all isomorphism classes of connected reductive algebraic groups of rank  $\leq n$ , and fix a group in every class. The obtained set of groups  $\mathcal{R}_n$  is finite. Therefore,

$$J_{\leq n} := \sup_{R \in \mathcal{R}_n} J_R \quad (18)$$

is a positive integer.

**Theorem 15.** *Let  $X$  be an irreducible variety of dimension  $n$ . Then every finite subgroup  $G$  of every connected affine algebraic subgroup of  $\text{Bir}(X)$  has a normal abelian subgroup whose index in  $G$  is at most  $J_{\leq n}$ .*

*Proof.* Let  $L$  be a connected affine algebraic subgroup of  $\text{Bir}(X)$  containing  $G$ . Being finite,  $G$  is reductive. Let  $R$  be a maximal reductive subgroup of  $L$  containing  $G$ . Then  $L$  is a semidirect product of  $R$  and the unipotent radical of  $L$ , see [Mo56, Thm. 7.1]. Therefore,  $R$  is connected because  $L$  is. Faithfulness of the action  $R$  acts on  $X$  yields that  $\text{rk } R \leq \dim X$ , see, e.g., [Po11, Lemma 2.4]. The claim then follows from (18), Theorem 2, and Definition 1.  $\square$

Theorem 15 and Definition 1 imply

**Corollary 11.** *Let  $X$  be an irreducible variety of dimension  $n$  such that  $\text{Bir}(X)$*

*(resp.  $\text{Aut}(X)$ ) is non-Jordan. Then for every integer  $d > J_{\leq n}$ , there is a finite subgroup  $G$  of  $\text{Bir}(X)$  (resp.  $\text{Aut}(X)$ ) with the properties:*

- (i)  $G$  does not lie in any connected affine algebraic subgroup of  $\text{Bir}(X)$  (resp.  $\text{Aut}(X)$ );
- (ii) for any abelian normal subgroup of  $G$ , its index in  $G$  is  $\geq d$ .

**Corollary 12.** *If  $\text{Cr}_n$  (resp.  $\text{Aut}(\mathbf{A}^n)$ ) is non-Jordan, then for every integer  $d > J_{\leq n}$ , there is a finite subgroup  $G$  of  $\text{Cr}_n$  (resp.  $\text{Aut}(\mathbf{A}^n)$ ) with the properties:*

- (i) the action of  $G$  on  $\mathbf{A}^n$  is nonlinearizable;
- (ii) for any abelian normal subgroup of  $G$ , its index in  $G$  is  $\geq d$ .

*Proof.* This follows from Corollary 11 because  $\mathbf{GL}_n(k)$  is a connected affine algebraic subgroup of  $\text{Aut}(\mathbf{A}^n)$  and nonlinearizability of the action of  $G$  on



$\mathbf{A}^n$  means that  $G$  is not contained in a subgroup of  $\mathrm{Cr}_n$  (resp.  $\mathrm{Aut}(\mathbf{A}^n)$ ) conjugate to  $\mathbf{GL}_n(k)$ .  $\square$

2.3.5. *Recent developments.* The initiated in [Po10] line of research of Jordaness of  $\mathrm{Aut}(X)$  and  $\mathrm{Bir}(X)$  for algebraic varieties  $X$  has generated interest of algebraic geometers in Moscow among whom I have promoted it, and led to a further progress in Problem B (hence A as well) in papers [Za10], [PS13<sub>1</sub>], [PS13<sub>2</sub>]. In [Za10], the earliest of them, the examples of non-Jordan groups  $\mathrm{Bir}(X)$  only known to date (October 2013) have been constructed (see Theorem 13 above). Below are formulated the results obtained in [PS13<sub>1</sub>], [PS13<sub>2</sub>]. Some of them are conditional, valid under the assumption that the following general conjecture by A. Borisov, V. Alexeev, and L. Borisov holds true:

*BAB Conjecture.* All Fano varieties of a fixed dimension  $n$  and with terminal singularities are contained in a finite number of algebraic families.

**Theorem 16** ([PS13<sub>1</sub>, Thm. 1.8]). *If the BAB Conjecture holds true in dimension  $n$ , then, for every rationally connected  $n$ -dimensional variety  $X$ , the group  $\mathrm{Bir}(X)$  is Jordan and, moreover,  $J_{\mathrm{Bir}(X)} \leq u_n$  for a number  $u_n$  depending only on  $n$ .*

Since for  $n = 3$  the BAB Conjecture is proved [KMMT00], this yields

**Corollary 13** ([PS13<sub>1</sub>, Cor. 1.9]). *The space Cremona group  $\mathrm{Cr}_3$  is Jordan.*

**Proposition 1** ([PS13<sub>1</sub>, Prop. 1.11]).  $u_3 \leq (25920 \cdot 20736)^{20736}$ .

The pivotal idea of the proof of Theorem 16 is to use a technically refined form of the “fixed-point method” described in Subsection 2.2.3.

**Theorem 17** ([PS13<sub>2</sub>, Thm. 1.8]). *Let  $X$  be an irreducible smooth proper  $n$ -dimensional variety.*

- (i) *The group  $\mathrm{Bir}(X)$  is Jordan in either of the cases:*
  - (a)  *$X$  is non-uniruled;*
  - (b) *the BAB Conjecture holds true in dimension  $n$  and the irregularity of  $X$  (i.e., the dimension of its Picard variety) is 0.*
- (ii) *If  $X$  is non-uniruled and its irregularity is 0, then the group  $\mathrm{Bir}(X)$  is bounded (see Definition 2).*

### 3. APPENDIX: PROBLEMS

Below I add a few additional problems to those which have already been formulated above (Problems A and B in Subsection 2.1, and Questions 1, 2).

3.1.  **$\mathrm{Cr}_n$ -conjugacy of finite subgroups of  $\mathbf{GL}_n(k)$ .** Below  $\mathbf{GL}_n(k)$  is identified in the standard way with the subgroup of  $\mathrm{Cr}_n$ , which, in turn, is identified with the subgroup of  $\mathrm{Cr}_m$  for any  $m = n + 1, n + 2, \dots, \infty$  (cf. [Po11, Sect. 1] or [Po12<sub>1</sub>, Sect. 1]).

**Question 3.** Consider the following properties of two finite subgroups  $A$  and  $B$  of  $\mathbf{GL}_n(k)$ :

- (i)  $A$  and  $B$  are isomorphic,
- (ii)  $A$  and  $B$  are conjugate in  $\mathrm{Cr}_n$ .

Does (i) imply (ii)?

*Comments.*

1. Direct verification based on the classification in [DI09] shows that the answer is affirmative for  $n \leq 2$ .

2. By [Po12<sub>1</sub>, Cor. 5], if  $A$  and  $B$  are abelian, then the answer is affirmative for every  $n$ .

3. If  $A$  and  $B$  are isomorphic, then they are conjugate in  $\mathrm{Cr}_{2n}$ . This is the corollary of the following stronger statement:

**Proposition 2.** *For any finite group  $G$  and any injective homomorphisms*

$$G \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} \mathbf{GL}_n(k), \quad (19)$$

*there exists an element  $\varphi \in \mathrm{Cr}_{2n}$  such that  $\alpha_1 = \mathrm{Int}(\varphi) \circ \alpha_2$ .*

*Proof.* Every element  $g \in \mathbf{GL}_n(k)$  is a linear transformation  $x \mapsto g \cdot x$  of  $\mathbf{A}^n$  (with respect to the standard structure of  $k$ -linear space on  $\mathbf{A}^n$ ). The injections  $\alpha_1$  and  $\alpha_2$  determine two faithful linear actions of  $G$  on  $\mathbf{A}^n$ : the  $i$ th action ( $i = 1, 2$ ) maps  $(g, x) \in G \times \mathbf{A}^n$  to  $\alpha_i(g) \cdot x$ . Consider the product of these actions, i.e., the action of  $G$  on  $\mathbf{A}^n \times \mathbf{A}^n$  defined by

$$G \times \mathbf{A}^n \times \mathbf{A}^n \rightarrow \mathbf{A}^n \times \mathbf{A}^n, \quad (g, x, y) \mapsto (\alpha_1(g) \cdot x, \alpha_2(g) \cdot y). \quad (20)$$

The natural projection of  $\mathbf{A}^n \times \mathbf{A}^n \rightarrow \mathbf{A}^n$  to the  $i$ th factor is  $G$ -equivariant. By classical Speiser's Lemma (see [LPR06, Lemma 2.12] and references therein), this implies that  $\mathbf{A}^n \times \mathbf{A}^n$  endowed with  $G$ -action (20) is  $G$ -equivariantly birationally isomorphic to  $\mathbf{A}^n \times \mathbf{A}^n$  endowed with the  $G$ -action via the  $i$ th factor by means of  $\alpha_i$ . Therefore,  $\mathbf{A}^n \times \mathbf{A}^n$  endowed with the  $G$ -action via the first factor by means of  $\alpha_1$  is  $G$ -equivariantly birationally isomorphic to  $\mathbf{A}^n \times \mathbf{A}^n$  endowed with the  $G$ -action via the second factor by means of  $\alpha_2$ ; whence the claim.  $\square$

**Remark 8.** In general, it is impossible to replace  $\mathrm{Cr}_{2n}$  by  $\mathrm{Cr}_n$  in Proposition 8. Indeed, in [RY02] one finds the examples of finite abelian groups  $G$  and embeddings (19) such that  $\alpha_1 \notin \mathrm{Int}(\mathrm{Cr}_n) \circ \alpha_2$ . However, since the images of these embeddings are isomorphic finite abelian subgroups of  $\mathbf{GL}_n(k)$ , by [Po12<sub>1</sub>, Cor. 5] these images are conjugate in  $\mathrm{Cr}_n$ .

**3.2. Torsion primes.** Let  $X$  be an irreducible variety. The following definition is based on the fact that the notion of algebraic torus in  $\mathrm{Bir}(X)$  makes sense.

**Definition 4** ([Po12<sub>1</sub>, Sect. 8]). Let  $G$  be a subgroup of  $\mathrm{Bir}(X)$ . A prime integer  $p$  is called a *torsion number* of  $G$  if there exists a finite abelian  $p$ -subgroup of  $G$  that does not lie in any torus of  $G$ .

Let  $\text{Tors}(G)$  be the set of all torsion primes of  $G$ . If  $G$  is a connected reductive algebraic subgroup of  $\text{Bir}(X)$ , this set coincides with that of the torsion primes of algebraic group  $G$  in the sense of classical definition, cf., e.g., [Se00, 1.3].

**Question 4** ([Po12<sub>1</sub>, Quest. 3]). What are, explicitly,

$$\text{Tors}(\text{Cr}_n), \quad \text{Tors}(\text{Aut } \mathbf{A}^n), \quad \text{Tors}(\text{Aut}^* \mathbf{A}^n), \quad n = 3, 4, \dots, \infty$$

where  $\text{Aut}^* \mathbf{A}^n$  is the group of those automorphisms of  $\mathbf{A}^n$  that preserve the volume form  $dx_1 \wedge \dots \wedge dx_n$  on  $\mathbf{A}^n$  (here  $x_1, \dots, x_n$  are the standard coordinate functions on  $\mathbf{A}^n$ ), cf. [Po12<sub>1</sub>, §2]?

*Comments.* By [Po12<sub>1</sub>, Sect. 8],

$$\begin{aligned} \text{Tors}(\text{Cr}_1) &= \{2\}, \\ \text{Tors}(\text{Cr}_2) &= \{2, 3, 5\} \quad (\text{this coincides with } \text{Tors}(E_8)), \\ \text{Tors}(\text{Cr}_n) &\supseteq \{2, 3\} \quad \text{for any } n \geq 3, \\ \text{Tors}(\text{Aut } \mathbf{A}^n) &= \text{Tors}(\text{Aut}^* \mathbf{A}^n) = \emptyset \quad \text{for } n \leq 2. \end{aligned}$$

**Question 5** ([Po12<sub>1</sub>, Quest. 4]). What is the minimal  $n$  such that 7 lies in one of the sets  $\text{Tors}(\text{Cr}_n)$ ,  $\text{Tors}(\text{Aut } \mathbf{A}^n)$ ,  $\text{Tors}(\text{Aut}^* \mathbf{A}^n)$ ?

**Question 6** ([Po12<sub>1</sub>, Quest. 5]). Are these sets finite?

**Question 7.** Are the sets

$$\bigcup_{n \geq 1} \text{Tors}(\text{Cr}_n), \quad \bigcup_{n \geq 1} \text{Tors}(\text{Aut } \mathbf{A}^n), \quad \bigcup_{n \geq 1} \text{Tors}(\text{Aut}^* \mathbf{A}^n)$$

finite?

**3.3. Embeddability in  $\text{Bir}(X)$ .** Not every group  $G$  is embeddable in  $\text{Bir}(X)$  for some  $X$ . For instance, by [Co13<sub>2</sub>, Thm. 1.2], if  $G$  is finitely generated, its embeddability in  $\text{Bir}(X)$  implies that  $G$  has a solvable word problem. Another example: by [Ca12],  $\mathbf{PGL}_\infty(k)$  is not embeddable in  $\text{Bir}(X)$  for  $k = \mathbf{C}$  (I thank S. Cantat who informed me in [Ca13] about these examples).

If  $\text{Bir}(X)$  is Jordan, then by Example 5 and Theorem 3(1)(i),  $\mathcal{N}$  is not embeddable in  $\text{Bir}(X)$ . Hence, by Theorem 17(i)(a),  $\mathcal{N}$  is not embeddable in  $\text{Bir}(X)$  for any non-uniruled  $X$ .

**Conjecture 1.** The finitely generated group  $\mathcal{N}$  defined in Example 5 is not embeddable in  $\text{Bir}(X)$  for every irreducible variety  $X$ .

Since  $\mathcal{N}$  contains  $\text{Sym}_n$  for every  $n$ , and every finite group can be embedded in  $\text{Sym}_n$  for an appropriate  $n$ , the existence of an irreducible variety  $X$  for which  $\text{Bir}(X)$  contains an isomorphic copy of  $\mathcal{N}$  implies that  $\text{Bir}(X)$  contains an isomorphic copy of every finite group and, in particular, every simple finite group. Therefore, Conjecture 1 follows from the affirmative answer to

**Question 8.** Let  $X$  be an irreducible variety. Is any set of pairwise nonisomorphic simple nonabelian finite subgroups of  $\text{Bir}(X)$  finite?

The affirmative answer looks likely. At this writing (October 2013) about this question I know the following:

**Proposition 3.** *If  $\dim X \leq 2$ , then the answer to Question 8 is affirmative.*

*Proof.* The claim immediately follows from Theorems 14 and 4 if  $X$  is not birationally isomorphic to  $\mathbf{P}^1 \times B$ , where  $B$  is an elliptic curve. For  $X = \mathbf{P}^1 \times B$  the claim follows from (17) and Theorem 4 because  $\text{Aut}(B)$  and  $\text{Bir}_B(X)$  are Jordan groups.  $\square$

By Theorems 16, 17 and by [KMMT00], the answer to Question 8 is affirmative also in each of the following cases:

- (i)  $X$  is non-uniruled;
- (ii)  $X$  is rationally connected or smooth proper with irregularity 0, and
  - (a) either  $\dim X = 3$  or
  - (b)  $\dim X > 3$  and the BAB Conjecture holds true in dimension  $\dim X$ .

Note that if  $X$  and  $\text{Bir}(X)$  in Question 8 are replaced, respectively, by a connected smooth topological manifold  $M$  and  $\text{Diff}(M)$ , then by Theorem 6, for a noncompact  $M$ , the answer, in general, is negative. But for a compact  $M$  a finiteness theorem [Po13, Thm. 2] holds.

**3.4. Contractions.** Developing the classical line of research, in recent years were growing activities aimed at description of finite subgroups of  $\text{Bir}(X)$  for various  $X$ ; the case of rational  $X$  (i.e., that of the Cremona group  $\text{Bir}(X)$ ) was, probably, most actively explored with culmination in the classification of finite subgroups of  $\text{Cr}_2$ , [DI09]. In these studies, all the classified groups appear in the corresponding lists on equal footing. However, in fact, some of them are “more basic” than the others because the latter may be obtained from the former by a certain standard construction. Given this, it is natural to pose the problem of describing these “basic” groups.

Namely, let  $X_1$  and  $X_2$  be the irreducible varieties and let  $G_i \subset \text{Bir}(X_i)$ ,  $i = 1, 2$ , be the subgroups isomorphic to a finite group  $G$ . Assume that fixing the isomorphisms  $G \rightarrow G_i$ ,  $i = 1, 2$ , we obtain the rational actions of  $G$  on  $X_1$  and  $X_2$  such that there is a  $G$ -equivariant rational dominant map  $\varphi: X_1 \dashrightarrow X_2$ . Let  $\pi_{X_i}: X_i \dashrightarrow X_i/G$ ,  $i = 1, 2$  be the rational quotients (see, e.g. [Po11, Sect. 1]) and let  $\varphi_G: X_1/G \dashrightarrow X_2/G$  be the dominant rational map induced by  $\varphi$ . Then the following holds (see, e.g. [Re00, Sect. 2.6]):

- (1) The appearing commutative diagram

$$\begin{array}{ccc}
 X_1 & \dashrightarrow^{\varphi} & X_2 \\
 \pi_{X_1} \downarrow & & \downarrow \pi_{X_2} \\
 X_1/G & \dashrightarrow^{\varphi_G} & X_2/G
 \end{array} \tag{21}$$

is, in fact, cartesian, i.e.,  $\pi_{X_1}: X_1 \dashrightarrow X_1/G$  is obtained from  $\pi_{X_2}: X_2 \dashrightarrow X_2/G$  by the base change  $\varphi_G$ . In particular,  $X_1$  is birationally  $G$ -isomorphic

to

$$X_2 \times_{X_2/G} (X_1/G) := \overline{\{(x, y) \in X_2 \times (X_1/G) \mid \pi_{X_2}(x) = \varphi_G(y)\}}.$$

(2) For every irreducible variety  $Z$  and every dominant rational map  $\beta: Z \rightarrow X_2/G$  such that  $X_2 \times_{X_2/G} Z$  is irreducible, the variety  $X_2 \times_{X_2/G} Z$  inherits via  $X_2$  a faithful rational action of  $G$  such that one obtains commutative diagram (21) with  $X_1 = X_2 \times_{X_2/G} Z$ ,  $\varphi_G = \beta$ , and  $\varphi = \text{pr}_1$ .

If such a  $\varphi$  exists, we say that  $G_1$  is *induced* from  $G_2$  by a base change. The latter is called *trivial* if  $\varphi$  is a birational isomorphism. If a finite subgroup  $G$  of  $\text{Bir}(X)$  is not induced by a nontrivial base change, we say that  $G$  is *incompressible*.

**Example 14.** The standard embedding  $\text{Cr}_n \hookrightarrow \text{Cr}_{n+1}$  permits to consider the finite subgroups of  $\text{Cr}_n$  as that of  $\text{Cr}_{n+1}$ . Every finite subgroup of  $\text{Cr}_{n+1}$  obtained this way is induced by the nontrivial base change determined by the projection  $\mathbf{A}^{n+1} \rightarrow \mathbf{A}^n$ ,  $(a_1, \dots, a_n, a_{n+1}) \mapsto (a_1, \dots, a_n, a_{n+1})$ .  $\square$

**Example 15.** This is Example 6 in [Re04]. Let  $G$  be a finite group that does not embed in  $\text{Bir}(Z)$  for any curve  $Z$  of genus  $\leq 1$  (for instance,  $G = \text{Sym}_5$ ) and let  $X$  be a smooth projective curve of minimal possible genus such that  $G$  is isomorphic to a subgroup of  $\text{Aut}(X)$ . Then this subgroup of  $\text{Bir}(X)$  is incompressible.<sup>1</sup>  $\square$

**Example 16.** By Example 5 in [Re04], a finite cyclic subgroup of order  $\geq 2$  in  $\text{Bir}(X)$  is never incompressible.  $\square$

**Example 17.** Consider two rational actions of  $G := \text{Sym}_3 \times \mathbb{Z}/2\mathbb{Z}$  on  $\mathbf{A}^3$ . The subgroup  $\text{Sym}_3$  acts by natural permuting the coordinates in both cases. The nontrivial element of  $\mathbb{Z}/2\mathbb{Z}$  acts by  $(a_1, a_2, a_3) \mapsto (-a_1, -a_2, -a_3)$  in the first case and by  $(a_1, a_2, a_3) \mapsto (a_1^{-1}, a_2^{-1}, a_3^{-1})$  in the second. The surfaces

$$P := \{(a_1, a_2, a_3) \in \mathbf{A}^3 \mid a_1 + a_2 + a_3 = 0\},$$

$$T := \{(a_1, a_2, a_3) \in \mathbf{A}^3 \mid a_1 a_2 a_3 = 1\}$$

are  $G$ -stable in, resp., the first and the second case. Since  $P$  and  $T$  are rational, these actions of  $G$  on  $P$  and  $T$  determine, up to conjugacy, resp., the subgroups  $G_P$  and  $G_T$  of  $\text{Cr}_2$ , both isomorphic to  $G$ . By [Is03] (see also [LPR06], [LPR07]), these subgroups are not conjugate in  $\text{Cr}_2$ . However, by [LPR07, Sect. 5],  $G_T$  is induced from  $G_P$  by a nontrivial base change (of degree 2).  $\square$

In fact, Example 17 is a special case (related to the simple algebraic group  $\mathbf{G}_2$ ) of the following

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<sup>1</sup>The proof in [Re04] should be corrected as follows. Assume that there is a faithful action of  $G$  of a smooth projective curve  $Y$  and a dominant  $G$ -equivariant morphism  $\varphi: X \rightarrow Y$  of degree  $n > 1$ . By the construction,  $X$  and  $Y$  have the same genus  $g > 1$ , and the Hurwitz formula yields that the number of branch points of  $\varphi$  (counted with positive multiplicities) is the integer  $(n - 1)(2 - 2g)$ . But the latter is negative, — a contradiction.

**Example 18.** Let  $G$  be a connected reductive algebraic group. Recall [LPR06, Def.1.5] that  $G$  is called a Cayley group if there is a birational isomorphism of  $\lambda: G \dashrightarrow \text{Lie}(G)$ , where  $\text{Lie}(G)$  is the Lie algebra of  $G$ , equivariant with respect to the conjugating and adjoint actions of  $G$  on the underlying varieties of  $G$  and  $\text{Lie}(G)$ , respectively, i.e., such that

$$\lambda(gXg^{-1}) = \text{Ad}_G g(\lambda(X)) \quad (22)$$

if  $g$  and  $X \in G$  and both sides of (22) are defined.

Fix a maximal torus  $T$  of  $G$  and consider the natural actions of the Weyl group  $W = N_G(T)/T$  on  $T$  and on  $\mathfrak{t} := \text{Lie}(T)$ . Since these actions are faithful and  $T$  and  $\mathfrak{t}$  are rational varieties, this determines, up to conjugacy, two embeddings of  $W$  in  $\text{Cr}_r$ , where  $r = \dim T$ . Let  $W_T$  and  $W_{\mathfrak{t}}$  be the images of these embeddings. By [LPR06, Lemma 3.5(a) and Sect. 1.5], if  $G$  is not Cayley and  $W$  has no outer automorphisms, then  $W_T$  and  $W_{\mathfrak{t}}$  are not conjugate in  $\text{Cr}_r$ . On the other hand, by [LPR06, Lemma 10.3],  $W_T$  is induced from  $W_{\mathfrak{t}}$  by a (nontrivial) base change (see also Lemma 5 below).

This yields, for arbitrary  $n$ , the examples of isomorphic nonconjugate finite subgroups of  $\text{Cr}_n$  one of which is induced from the other by a nontrivial base change. For instance, if  $G = \mathbf{SL}_{n+1}$ , then  $r = n$  and  $W = \text{Sym}_n$ . Since, by [LPR06, Thm. 1.31],  $\mathbf{SL}_{n+1}$  is not Cayley for  $n \geq 3$  and  $\text{Sym}_n$  has no outer automorphisms for  $n \neq 6$ , the above construction yields for these  $n$  two nonconjugate subgroups of  $\text{Cr}_n$  isomorphic to  $\text{Sym}_n$ , one of which is induced from the other by a nontrivial base change.  $\square$

The following gives a general way of constructing two finite subgroups of  $\text{Cr}_n$  one of which is induced from the other by a base change.

Consider an  $n$ -dimensional irreducible nonsingular variety  $X$  and a finite subgroup  $G$  of  $\text{Aut}(X)$ . Suppose that  $x \in X$  is a fixed point of  $G$ . By Lemma 4, the induced action of  $G$  on the tangent space of  $X$  at  $x$  is faithful. Therefore this action determines, up to conjugacy, a subgroup  $G_1$  of  $\text{Cr}_n$  isomorphic to  $G$ . On the other hand, if  $X$  is rational, the action of  $G$  on  $X$  determines, up to conjugacy, another subgroup  $G_2$  of  $\text{Cr}_n$  isomorphic to  $G$ .

**Lemma 5.**  $G_2$  is induced from  $G_1$  by a base change.

*Proof.* By Lemma 3 we may assume that  $X$  is affine, in which case the claim follows from [LPR06, Lemma 10.3].  $\square$

**Corollary 14.** Let  $X$  be a nonrational irreducible variety and let  $G$  be an incompressible finite subgroup of  $\text{Aut}(X)$ . Then  $X^G = \emptyset$ .

**Question 9.** Which finite subgroups of  $\text{Cr}_2$  are incompressible?

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