Integrated Quantile Functions: Properties and Applications

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June 12, 2017

Preface

This presentation is based on some results of our joint work with prof. A. A. Gushchin (Steklov Mathematical Institute, National Research University Higher School of Economics).

Introduction — 1

Integrated distribution and quantile functions or simple transformations of them play an important role in probability theory, mathematical statistics, and their applications such as insurance, finance, economics etc.

Introduction — 2

In this presentation we suggest a systematic exposition of basic properties of integrated distribution and quantile functions. In particular, we define integrated distribution and quantile functions for a random variable X without any restrictions on the distribution of X. At the same time, as we know, in the literature integrated distribution and quantile functions appear under additional assumptions: $\mathbb{E}[X^+] < \infty$, $\mathbb{E}[X^-] < \infty$ or $\mathbb{E}|X| < \infty$.

Introduction — 3

In this research we obtain the characteristic properties of integrated distribution and quantile functions. Moreover, we express such important notions of probability theory as **tightness**, **uniform integrability** and **weak convergence** in terms of integrated quantile functions. In addition, we provide an example of the proof of a known probability inequality using integrated quantile functions technique.

Distribution function

As usual, the ***distribution function of a random variable*** X is defined by

$$F_X(x) := \mathsf{P}(\{X \leq x\}), \quad x \in \mathbb{R}.$$

Integrated distribution function (IDF)

Definition

The ***integrated distribution function of a random variable*** X is defined by

$$\mathrm{J}_X(x) := \int_0^x F_X(t)\,dt, \qquad x \in \mathbb{R},$$

with convention: $\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$, if $b < a$.

Properties of IDF

Theorem

An integrated distribution function J_X *has the following properties:*

(i) $\mathrm{J}_X(0) = 0$,

(ii) J_X *is convex, increasing and finite everywhere on* $\mathbb{R}$,

(iii) $\lim\limits_{x \to -\infty} \mathrm{J}_X(x) = -\mathrm{E}[X^-]$ *and* $\lim\limits_{x \to +\infty} (x - \mathrm{J}_X(x)) = \mathrm{E}[X^+]$,

(iv) $\lim\limits_{x \to -\infty} \frac{\mathrm{J}_X(x)}{x} = 0$ *and* $\lim\limits_{x \to +\infty} \frac{\mathrm{J}_X(x)}{x} = 1$,

(v) *the subdifferential of* J_X *satisfies*

$$\partial \mathrm{J}_X(x) = [F_X(x-0);\ F_X(x)], \quad x \in \mathbb{R},$$

in particular, $(\mathrm{J}_X)'_-(x) = F_X(x-0)$ *and* $(\mathrm{J}_X)'_+(x) = F_X(x)$.

A typical plot of IDF

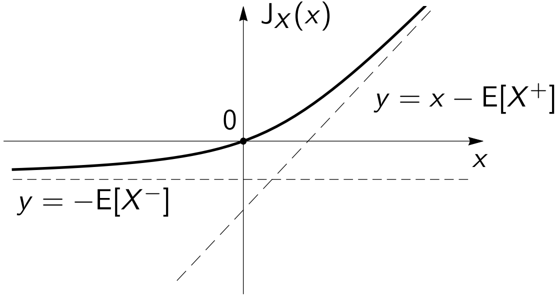

Figure: A typical plot of IDF under assumption $E[X^-] < \infty$, $E[X^+] < \infty$.

A characterisation of IDF

Theorem

If $J\colon \mathbb{R} \to \mathbb{R}$, $J(0) = 0$, *is a convex function satisfying*

$$\lim_{x \to -\infty} \frac{J(x)}{x} = 0 \quad \textit{and} \quad \lim_{x \to +\infty} \frac{J(x)}{x} = 1,$$

then there exists on some probability space a random variable X *for which* $\mathrm{J}_X = J$.

Quantile function

We call a ***quantile function*** of a random variable X every function $q_X : (0;\, 1) \to \mathbb{R}$ satisfying

$$\forall u \in (0;\, 1):\ \ F_X\big(q_X(u) - 0\big) \leq u \leq F_X\big(q_X(u)\big).$$

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The functions q_X^L and q_X^R defined by

$$q_X^L(u) := \inf\big\{x \in \mathbb{R}\colon F_X(x) \ge u\big\},$$

$$q_X^R(u) := \inf\big\{x \in \mathbb{R}\colon F_X(x) > u\big\},$$

are called the ***lower (left)*** and ***upper (right)*** quantile functions of X.

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are called the ***lower (left)*** and ***upper (right)*** quantile functions of X.

It follows directly from the definitions that lower and upper quantile functions of X are quantile functions of X, and for any quantile function q_X we always have

$$q_X^L(u) \le q_X(u) \le q_X^R(u), \qquad u \in (0;\, 1).$$

Legendre–Young–Fenchel transformation

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$. The ***Fenchel transformation of*** f or the ***conjugate function of*** f is the function $f^*\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by the following rule:

$$f^*(y) := \sup_{x\in\mathbb{R}^n} \big\{\langle x, y\rangle - f(x)\big\}.$$

Integrated quantile function (IQF)

Definition

The Fenchel transform of the integrated distribution function of a random variable X

$$\mathrm{K}_X(u) := \sup_{x \in \mathbb{R}} \{xu - \mathrm{J}_X(x)\}, \quad u \in \mathbb{R},$$

is called the ***integrated quantile function*** of X.

The statement (v) of the next theorem explains the meaning of the term 'integrated quantile function'.

Properties of IQF — 1

Theorem

An integrated quantile function K_X *possesses the following properties:*

(i) *the function* K_X *is convex and lower semicontinuous,*

(ii) *it takes finite values on* $(0;\,1)$ *and equals* $+\infty$ *outside* $[0;\,1]$,

(iii) *the Fenchel transform of* K_X *is* J_X, *i. e. for any* $x \in \mathbb{R}$,

$$\sup_{u \in \mathbb{R}} \{xu - \mathrm{K}_X(u)\} = \mathrm{J}_X(x),$$

(iv) $\min_{u \in \mathbb{R}} \mathrm{K}_X(u) = 0$ *and* $\mathrm{K}_X^{-1}(0) = [F_X(0-0);\, F_X(0)]$,

(v) *for every* $u \in [0;\,1]$,

$$\mathrm{K}_X(u) = \int_{u_0}^{u} q_X(s)\, ds,$$

where u_0 *is any zero of* K_X,

Properties of IQF — 2

Theorem

(vi) $\mathsf{K}_X(0) = \mathsf{E}[X^-]$ *and* $\mathsf{K}_X(1) = \mathsf{E}[X^+]$,

(vii) *the subdifferential of* K_X *satisfies*

$$\partial \mathsf{K}_X(u) = [q_X^L(u);\, q_X^R(u)], \qquad u \in (0;\, 1),$$

in particular, $(\mathsf{K}_X)'_-(u) = q_X^L(u)$ *and* $(\mathsf{K}_X)'_+(u) = q_X^R(u)$.

A typical plot of IQF

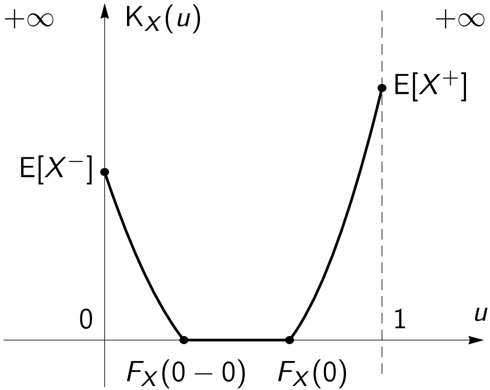

Figure: A typical plot of IQF.

A characterisation of IQF

Theorem

If a convex lower semicontinuous function $K\colon \mathbb{R} \to [0; +\infty]$ *satisfies*

$$(0; 1) \subseteq \{u \in \mathbb{R} \colon K(u) < +\infty\} \subseteq [0; 1]$$

and there is $u_0 \in [0; 1]$ *such that* $K(u_0) = 0$*, then there exists a random variable* X *on some probability space such that* $\mathrm{K}_X = K$*.*

IQF and the convex orders — 1

Let X and Y be random variables with finite means. Then we say that

- X is less than Y in ***convex order*** ($X \leq_{cx} Y$) if $\mathsf{E}[\varphi(X)] \leq \mathsf{E}[\varphi(Y)]$ for any real convex function φ such that both expectations exist,
- X is less than Y in ***increasing convex order*** ($X \leq_{icx} Y$) if $\mathsf{E}[\varphi(X)] \leq \mathsf{E}[\varphi(Y)]$ for any increasing real convex function φ such that both expectations exist.

IQF and the convex orders — 2

Assuming that $\mathsf{K}_X(1) = \mathsf{E}[X^+] < \infty$ let us introduce a ***shifted integrated quantile function***:

$$\mathsf{K}_X^{[1]}(u) := \mathsf{K}_X(u) - \mathsf{K}_X(1), \quad u \in [0;\, 1].$$

Theorem (convex order criterion)

Let X and Y be random variables with finite means.

(i) *$X \leq_{cx} Y$ if and only if $\mathsf{K}_X^{[1]}(u) \geq \mathsf{K}_Y^{[1]}(u)$ for all $u \in [0;\, 1]$ and $\mathsf{K}_X^{[1]}(0) = \mathsf{K}_Y^{[1]}(0)$.*

(ii) *$X \leq_{icx} Y$ if and only if $\mathsf{K}_X^{[1]}(u) \geq \mathsf{K}_Y^{[1]}(u)$ for all $u \in [0;\, 1]$.*

IQF and tightness

Theorem

Let $\{X_\alpha\}$ be a family of random variables. The family of measures $\{\mathsf{P}_{X_\alpha}\}$ is tight if and only if the corresponding family of integrated quantile functions $\{\mathsf{K}_{X_\alpha}\}$ is uniformly bounded on each subinterval $[a, b] \subseteq (0; 1)$.

IQF and uniform integrability

Theorem

Let $\{X_\alpha\}$ be a family of random variables. The family of measures $\{\mathsf{P}_{X_\alpha}\}$ is uniformly integrable if and only if the corresponding family of integrated quantile functions $\{\mathsf{K}_{X_\alpha}\}$ is relatively compact in the space $C[0;1]$ of continuous functions with supremum norm.

IQF and weak convergence

Theorem

Let (X_n) be a sequence of random variables. The sequence (X_n) weakly converges if and only if the corresponding sequence of integrated quantile functions (K_{X_n}) converges uniformly on every subinterval $[a; b] \subseteq (0; 1)$.

Application of IQF — 1

Example

Let us study the sharp upper bound for the probability $\mathrm{P}(\{X \geq t\})$, where t is a fixed positive number and X ranges over the set of random variables with $\mathrm{E}[X] = 0$ and $\mathrm{D}(X) = \sigma^2 < \infty$.

Application of IQF — 1

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Let us fix a random variable X with $\mathsf{E}[X] = 0$ and $\mathsf{D}(X) = \sigma^2$, fix $t > 0$, and put $p := \mathsf{P}(\{X \geq t\})$.

Application of IQF — 2

Example

Recall a well-known property of quantile functions:

$$\forall t \in \mathbb{R} \quad \forall u \in (0;1): \quad u \geq F_X(t-0) \Leftrightarrow q_X^R(u) \geq t. \tag{1}$$

Application of IQF — 2

Example

Recall a well-known property of quantile functions:

$$\forall t \in \mathbb{R} \quad \forall u \in (0;1): \quad u \geq F_X(t-0) \Leftrightarrow q_X^R(u) \geq t. \qquad (1)$$

If $u \geq 1-p$, i. e. $u \geq F_X(t-0)$, then by (1) we have $q_X^R(u) \geq t$.

Application of IQF — 2

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Since q_X^R is an increasing function it follows that

$$t(1-u) \leq \int_u^1 q_X^R(s)\,ds = \mathrm{K}_X(1) - \mathrm{K}_X(u) = -\mathrm{K}_X^{[1]}(u).$$

Application of IQF — 2

Example

Recall a well-known property of quantile functions:

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$$t(1-u) \leq \int_u^1 q_X^R(s)\, ds = \mathrm{K}_X(1) - \mathrm{K}_X(u) = -\mathrm{K}_X^{[1]}(u).$$

Hence, for all $u \in [1-p;\, 1]$, we have

$$\mathrm{K}_X^{[1]}(u) \leq t(u-1). \tag{2}$$

Example

Further, let us introduce a discrete random variable Y such that

$$\mathrm{P}(\{Y = t\}) = p, \quad \mathrm{P}(\{Y = -\tfrac{tp}{1-p}\}) = 1 - p. \tag{3}$$

Application of IQF — 3

Example

Further, let us introduce a discrete random variable Y such that

$$\mathsf{P}(\{Y = t\}) = p, \quad \mathsf{P}(\{Y = -\tfrac{tp}{1-p}\}) = 1 - p. \tag{3}$$

It is easy to check that, for $u \in [0; 1]$,

$$\mathsf{K}_Y(u) = \begin{cases} -\frac{tp}{1-p}u + tp, & \text{if } u \in [0; 1-p], \\ t(u - (1-p)), & \text{if } u \in [1-p; 1], \end{cases}$$

Application of IQF — 3

Example

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and

$$\mathrm{K}_Y^{[1]}(u) = \mathrm{K}_Y(u) - \mathrm{K}_Y(1) = \begin{cases} -\frac{tp}{1-p}u, & \text{if } u \in [0; 1-p], \\ t(u-1), & \text{if } u \in [1-p; 1]. \end{cases} \tag{4}$$

Application of IQF — 4

Example

Now, let us remark that

$$\mathsf{K}_X^{[1]}(0) = \mathsf{K}_X(0) - \mathsf{K}_X(1) = \mathsf{E}[X^-] - \mathsf{E}[X^+] = -\mathsf{E}[X] = 0 = \mathsf{K}_Y^{[1]}(0),$$

Example

Now, let us remark that

$$\mathsf{K}_X^{[1]}(0) = \mathsf{K}_X(0) - \mathsf{K}_X(1) = \mathsf{E}[X^-] - \mathsf{E}[X^+] = -\mathsf{E}[X] = 0 = \mathsf{K}_Y^{[1]}(0),$$

and, for all $u \in [1-p; 1]$,

$$\mathsf{K}_X^{[1]}(u) \overset{(2)}{\leq} t(u-1) \overset{(4)}{=} \mathsf{K}_Y^{[1]}(u).$$

Example

Now, let us remark that

$$K_X^{[1]}(0) = K_X(0) - K_X(1) = E[X^-] - E[X^+] = -E[X] = 0 = K_Y^{[1]}(0),$$

and, for all $u \in [1-p; 1]$,

$$K_X^{[1]}(u) \overset{(2)}{\leq} t(u-1) \overset{(4)}{=} K_Y^{[1]}(u).$$

Moreover, for any $u \in [0;\, 1-p]$, we have $u = \alpha \cdot 0 + (1-\alpha) \cdot (1-p)$, where $\alpha \in [0;\, 1]$, and by convexity of $K_X^{[1]}$ and linearity of $K_Y^{[1]}$ on $[0;\, 1-p]$ it follows that

$$K_X^{[1]}(u) \leq \alpha \underbrace{K_X^{[1]}(0)}_{=0=K_Y^{[1]}(0)} + (1-\alpha) \underbrace{K_X^{[1]}(1-p)}_{\leq K_Y^{[1]}(1-p)} = K_Y^{[1]}(u).$$

Example

So, it is shown that $\mathrm{K}_X^{[1]}(0) = \mathrm{K}_Y^{[1]}(0)$ and $\mathrm{K}_X^{[1]}(u) \leq \mathrm{K}_Y^{[1]}(u)$, for all $u \in [0;\,1]$.

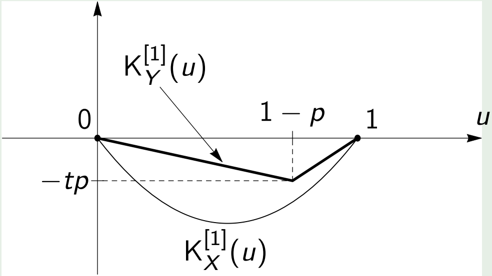

Figure: Plots of $\mathrm{K}_X^{[1]}$ and $\mathrm{K}_Y^{[1]}$.

Application of IQF — 6

Example

By a convex order criterion (see (i)) it follows from relations

$$\mathrm{K}_X^{[1]}(0) = \mathrm{K}_Y^{[1]}(0) \quad \text{and} \quad \mathrm{K}_X^{[1]}(u) \leq \mathrm{K}_Y^{[1]}(u), \quad \text{for all } u \in [0;\, 1],$$

that $Y \leq_{cx} X$.

Application of IQF — 6

Example

By a convex order criterion (see (i)) it follows from relations

$$\mathrm{K}_X^{[1]}(0) = \mathrm{K}_Y^{[1]}(0) \quad \text{and} \quad \mathrm{K}_X^{[1]}(u) \le \mathrm{K}_Y^{[1]}(u), \quad \text{for all } u \in [0;\, 1],$$

that $Y \le_{cx} X$. Hence, $\mathrm{E}[Y^2] \le \mathrm{E}[X^2]$ and

$$\tfrac{t^2 p}{1-p} = \mathrm{E}[Y^2] \le \mathrm{E}[X^2] = \mathrm{D}(X) = \sigma^2.$$

Example

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$$\tfrac{t^2 p}{1-p} = \mathrm{E}[Y^2] \le \mathrm{E}[X^2] = \mathrm{D}(X) = \sigma^2.$$

Resolving the last inequality with respect to $p = \mathrm{P}(\{X \ge t\})$ we get the required upper bound:

$$\mathrm{P}(\{X \ge t\}) \le \frac{\sigma^2}{\sigma^2 + t^2}. \tag{5}$$

Application of IQF — 7

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Let us prove that the estimate in (5) is **sharp**.

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Put $p = \frac{\sigma^2}{\sigma^2+t^2}$ in the definition (3) of a random variable Y.

Application of IQF — 7

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Put $p = \frac{\sigma^2}{\sigma^2+t^2}$ in the definition (3) of a random variable Y. Then we have:

$$\mathsf{P}(\{Y = t\}) = \frac{\sigma^2}{\sigma^2+t^2}, \quad \mathsf{P}(\{Y = -\frac{\sigma^2}{t}\}) = \frac{t^2}{\sigma^2+t^2},$$

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$$\mathrm{P}(\{Y = t\}) = \frac{\sigma^2}{\sigma^2+t^2}, \quad \mathrm{P}(\{Y = -\frac{\sigma^2}{t}\}) = \frac{t^2}{\sigma^2+t^2},$$

$$\mathrm{E}[Y] = 0, \quad \mathrm{D}(Y) = \sigma^2, \quad \mathrm{P}(\{Y \geq t\}) \overset{!}{=} \frac{\sigma^2}{\sigma^2+t^2}.$$

Application of IQF — 7

Example

Let us prove that the estimate in (5) is **sharp**.

Put $p = \frac{\sigma^2}{\sigma^2+t^2}$ in the definition (3) of a random variable Y. Then we have:

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$$\mathrm{E}[Y] = 0, \quad \mathrm{D}(Y) = \sigma^2, \quad \mathrm{P}(\{Y \geq t\}) \overset{!}{=} \frac{\sigma^2}{\sigma^2+t^2}.$$

So, the inequality (5) turned to equality, i. e. the estimate in (5) is sharp.

Thank you for your attention!