Singularity preserving maps on matrix algebras

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Notation

- $\mathbb{F}$ — an arbitrary field;
- $M_n(\mathbb{F})$ — the $n \times n$ matrix algebra over a field $\mathbb{F}$;
- $GL_n(\mathbb{F})$ — the set of invertible matrices;
- $\Omega_n(\mathbb{F})$ — the set of singular matrices.

Theorem (Frobenius, 1897)

If $T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is linear and preserves the determinant, i. e.,*

$$\det(T(A)) = \det(A) \text{ for all } A \in M_n(\mathbb{C}),$$

then T *is of the form*

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{C}) \quad \text{or} \quad T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{C}),$$

where $P, Q \in GL_n(\mathbb{C})$ *with* $\det(PQ) = 1$*.*

Let $\mathcal{Y}$ be a subset of $M_n(\mathbb{F})$. We say that a transformation $T\colon \mathcal{Y} \to M_n(\mathbb{F})$ is of a **standard form** if there exist non-singular matrices P, Q such that

$$T(A) = PAQ \quad \forall A \in \mathcal{Y} \quad \text{or} \quad T(A) = PA^tQ \quad \forall A \in \mathcal{Y}. \tag{1}$$

Theorem (Dieudonné, 1949)

Let $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear bijection. If T preserves the singularity, i. e.,

$$\det(A) = 0 \Rightarrow \det(T(A)) = 0,$$

then T is of the standard form (1).

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Theorem (Dolinar, Šemrl, 2002)

If $T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is surjective and satisfies*

$$\det(A+\lambda B) = \det(T(A)+\lambda T(B)) \quad \text{for all } A, B \in M_n(\mathbb{C}) \text{ and all } \lambda \in \mathbb{C}, \tag{2}$$

then T *is linear and hence is of the standard form (1) with* $\det(PQ) = 1$.

Let $\mathbb{F}$ be a field such that $|\mathbb{F}| > n$.

Theorem (Tan, Wang, 2003)

Let $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a transformation satisfying (2). Then T is of the standard form (1).

Theorem (Tan, Wang, 2003)

Let $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a surjective transformation satisfying

$$\det(A + \lambda_i B) = \det(T(A) + \lambda_i T(B)) \qquad \text{for all } A, B \in M_n \quad \text{and} \quad i = 1, 2$$

where $\lambda_i \in \mathbb{F} - \{0\}$ and $(\lambda_1/\lambda_2)^k \neq 1$ for $1 \leqslant k \leqslant n-2$. Then T is of the standard form (1).

Theorem (Costara, 2019)

Suppose $|\mathbb{F}| \geqslant n^2 + 1$. *Let* $T_1, T_2 : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *be maps, one of them being surjective, such that*

$$\det(T_1(A) + T_2(B)) = \det(A + B) \quad (A, B \in M_n(\mathbb{F})).$$

Then there exist $A_0 \in M_n(\mathbb{F})$ *and* $P, Q \in M_n(\mathbb{F})$ *satisfying* $\det(PQ) = 1$ *such that either*

$$T_1(A) = P\,(A + A_0)\,Q \quad \textit{and} \quad T_2(A) = P\,(A - A_0)\,Q \quad \forall A \in M_n(\mathbb{F})$$

or

$$T_1(A) = P\,(A + A_0)^t\,Q \quad \textit{and} \quad T_2(A) = P\,(A - A_0)^t\,Q \quad \forall A \in M_n(\mathbb{F}).$$

Theorem (Costara, 2019)

Let $\mathbb{F}$ *be a field with* $|\mathbb{F}| \geqslant n^2 + 1$, *and fix some nonzero element* $\lambda_0 \in \mathbb{F}$. *Let* $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *be a surjective map such that*

$$\det\left(T(A) + \lambda_0 T(B)\right) = \det\left(A + \lambda_0 B\right) \quad (A, B \in M_n(\mathbb{F}))$$

If $\lambda_0 = -1$, *there exist* $A_0 \in M_n(\mathbb{F})$ *and* $P, Q \in M_n(\mathbb{F})$ *satisfying* $\det(PQ) = 1$ *such that*

$$T(A) = P\left(A + A_0\right)Q \quad (A \in M_n(\mathbb{F})) \qquad \text{or} \qquad T(A) = P\left(A + A_0\right)^t Q \quad (A \in M_n(\mathbb{F}))$$

If $\lambda_0 \neq -1$, *then* T *is of the standard form (1).*

Main results

Let $\mathbb{F}$ be an algebraically closed field.

Theorem (Guterman, Maksaev, Promyslov, 2021+)

Suppose $\mathcal{Y} = GL_n(\mathbb{F})$ or $\mathcal{Y} = M_n(\mathbb{F})$, $T\colon \mathcal{Y} \to M_n(\mathbb{F})$ is a map satisfying the following conditions:

- *for all $A, B \in \mathcal{Y}$ and $\lambda \in \mathbb{F}$*

$$\det(A + \lambda B) = 0 \quad \Rightarrow \quad \det(T(A) + \lambda T(B)) = 0 \qquad (*)$$

- *the image of T contains at least one non-singular matrix.*

Then T is of the standard form (1).

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Then T *is of the standard form (1).*

Note that in the theorem above $\det(PQ)$ possibly differs from 1.

- It is enough to consider only such maps T, that $T(I) = I$.
- Indeed, if T satisfies $(*)$, then for every $R, S \in GL_n(\mathbb{F})$ map T' such that $T'(A) = R \cdot T(A) \cdot S$ also satisfies $(*)$.

Lemma

If $T(I) = I$ then T preserves determinant, i.e. $\det A = \det(T(A)) \quad \forall A \in GL_n(\mathbb{F})$.

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Sketch of proof

Identity matrix preservation

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$T : GL_n(\mathbb{F}) \to M_n(\mathbb{F})$

- The aim is to prove that:
  - $T(A+B) = T(A) + T(B) \quad \forall A, B \in GL_n(\mathbb{F})$ such that $A+B$ is non-singular;
  - $T(\alpha A) = \alpha T(A) \quad \forall A \in GL_n(\mathbb{F}), \alpha \in \mathbb{F}^*$.
- Then $T : GL_n(\mathbb{F}) \to M_n(\mathbb{F})$ can be extended by linearity on $M_n(\mathbb{F})$ in such way that $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ still preserves determinant.
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Sketch of proof

Considering matrices as vectors

- To prove linearity we used ideas of Victor Tan and Fei Wang. Matrices $A \in M_n(\mathbb{F})$ can be considered as vectors $\nu_A \in \mathbb{F}^{n^2}$. Then to prove linearity it is enough to show that

$$\nu_{T(A)} = X \cdot \nu_A \quad \text{for some matrix } X \in M_{n^2}(\mathbb{F}).$$

- But in our case instead of condition

$$\det(A + \lambda B) = \det(T(A) + \lambda T(B)) \quad \text{for all } A, B \in M_n(\mathbb{F}) \text{ and all } \lambda \in \mathbb{F} \tag{3}$$

we have

$$\det(A + \lambda B) = 0 \Rightarrow \det(T(A) + \lambda T(B)) = 0. \tag{$*$}$$

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- Note that if the polynomial has n distinct roots and $\det(A)=\det(T(A))$, then $(*)$ implies $\det(A+\lambda B)=\det(T(A)+\lambda T(B))$.
- Indeed, $\det(A+\lambda B)$ and $\det(T(A)+\lambda T(B))$ have the n common roots and coefficients of the term λ^0 are $\det(A)=\det(T(A))$.
- Thus it is enough to find for fixed A a matrix B such that $\det(A+\lambda B)$ has n distinct roots.
- This lead us to use some interesting properties of discriminant of polynomials.

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For $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ the theorem follows from the following lemma, which can be interesting by itself:

Lemma

Let $\mathbb{F}$ *be a field* $|\mathbb{F}| > n > 1$ *and* $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *denotes the map satisfying the following conditions:*

1) *for any matrices* A, B *the singularity of the matrix* $A + B$ *implies singularity of* $T(A) + T(B)$;
2) $T|_{GL_n(\mathbb{F})} = \mathrm{id}|_{GL_n(\mathbb{F})}$.

Then $T = \mathrm{id}$.

Thank you for your attention!

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