Loop homology and LS-category of moment-angle complexes for flag complexes.

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Objects of study

Let $\mathcal{K}$ be a simplicial complex on vertex set $[m]=\{1,\dots,m\}$. For any sequence of CW-pairs $(\underline{X},\underline{A})=((X_1,A_1),\dots,(X_m,A_m))$, consider the polyhedral product

$$(\underline{X},\underline{A})^{\mathcal{K}}:=\bigcup_{J\in\mathcal{K}}\Big(\prod_{i\in J}X_i\times\prod_{i\in[m]\setminus J}A_i\Big)\subset\prod_{i=1}^m X_i.$$

Such spaces naturally appear in seemingly unrelated areas of homotopy theory (higher Whitehead products), combinatorial algebra (graph-algebras), geometric group theory (right-angled Coxeter and Artin groups)...

In toric topology [BP15], the most important polyhedral products are moment-angle complexes $\mathcal{Z}_{\mathcal{K}}:=(D^2,S^1)^{\mathcal{K}}$ and Davis–Januszkiewicz spaces $\mathrm{DJ}(\mathcal{K}):=(\mathbb{C}P^\infty,\mathrm{pt})^{\mathcal{K}}$. Davis and Januszkiewicz [DJ91] used moment-angle complexes to classify quasitoric manifolds M over a simple polytope P: we always have $M\cong\mathcal{Z}_{\mathcal{K}}/H$, where $\mathcal{K}=\partial P^*$ and $H\subset\mathbb{T}^m\curvearrowright\mathcal{Z}_{\mathcal{K}}$ is a freely acting subtorus. We obtain $\mathrm{DJ}(\mathcal{K})$ as the homotopy quotient of both $\mathbb{T}^m\curvearrowright\mathcal{Z}_{\mathcal{K}}$ and $\mathbb{T}^m/H\curvearrowright M$. Cohomology of $\mathcal{Z}_{\mathcal{K}}$ and $\mathrm{DJ}(\mathcal{K})$ is well known:

Theorem [DJ91, BP15]. For any commutative ring $\mathbf{k}$ with unit, we have

$$H^*(\mathrm{DJ}(\mathcal{K});\mathbf{k})\cong\mathbf{k}[\mathcal{K}]$$

as rings and

$$H^*(\mathcal{Z}_{\mathcal{K}};\mathbf{k})\cong\mathrm{Tor}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k}[v_1,\dots,v_m],\mathbf{k})\cong\bigoplus_{J\subset[m]}\tilde{H}^{*-|J|-1}(\mathcal{K}_J;\mathbf{k})$$

as $\mathbf{k}$-modules, where

$$\mathbf{k}[\mathcal{K}]:=\mathbf{k}[v_1,\dots,v_m]/(\prod_{j\in J}v_j=0,\ J\notin\mathcal{K}),\quad \deg v_i=2.$$

Actually, $\mathbf{k}[\mathcal{K}]$ admits a $\mathbb{Z}^m_{\geq0}$-grading: $\deg v_j=2e_j=(0,\dots,0,2,0,\dots,0)\in\mathbb{Z}^m_{\geq0}$.

Loop homology and LS-category

Let X be a simply connected space and $\mathbf{k}$ be a commutative ring with unit. Then $H_*(\Omega X;\mathbf{k})$ is a connected cocommutative Hopf algebra with unit – the Pontryagin algebra of X. We study its presentations by generators and relations. The following fact is useful:

Theorem [Wa60]. Let A be a connected graded algebra over a field $\mathbf{k}$. Suppose that a homogeneous presentation

$$A\simeq T(a_1,\dots,a_N)/(r_1=\dots=r_M=0)$$

has no redundant generators or relations. Then

$$\mathrm{Tor}_1^A(\mathbf{k},\mathbf{k})\simeq\bigoplus_{i=1}^N\mathbf{k}\cdot a_i,\quad \mathrm{Tor}_2^A(\mathbf{k},\mathbf{k})\simeq\bigoplus_{j=1}^M\mathbf{k}\cdot r_j$$

as graded $\mathbf{k}$-modules.

Definition. The LS-category $\mathrm{cat}\,X$ of a topological space X is the smallest integer n such that there is an open covering $X=U_0\cup\dots\cup U_n$ with every $U_i\hookrightarrow X$ null-homotopic. (It is possible that $\mathrm{cat}\,X=+\infty$.)

A classical lower bound can be given in terms of the Milnor–Moore spectral sequence

$$E^2_{p,q}=\mathrm{Tor}_p^{H_*(\Omega X;\mathbf{k})}(\mathbf{k},\mathbf{k})_q\Rightarrow H_{p+q}(X;\mathbf{k}).$$

Theorem (M.Ginsburg). If $E^\infty_{p,q}\neq0$, then $\mathrm{cat}\,X\geq p$.

Hence it is useful to compute $\mathrm{Tor}_*^{H_*(\Omega X;\mathbf{k})}(\mathbf{k},\mathbf{k})$.

Approach of Panov and Ray

Panov and Ray noticed [PR08] that the fibration $\mathcal{Z}_{\mathcal{K}}\to\mathrm{DJ}(\mathcal{K})\to(\mathbb{C}P^\infty)^m$ admits a homotopy section after looping. We have an extension of Hopf algebras

$$1\to H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})\to H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})\to\Lambda[u_1,\dots,u_m]\to0,$$

so

$$H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})\simeq H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})\otimes\Lambda[u_1,\dots,u_m]$$

as left $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})$-modules (not as algebras!).

Definition. Simplicial complex $\mathcal{K}$ is flag if any set of pairwise connected vertices spans a simplex:

- $J\in\mathcal{K}$ whenever $\{i,j\}\in\mathcal{K},\ \forall i,j\in J$.

Theorem [PR08, Vy22].

1. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})\cong\mathrm{Ext}^*_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})$ as graded algebras:
$$H_n(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})\cong\bigoplus_{n=-i+2j}\mathrm{Ext}^i_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})_{2j}.$$
2. The "diagonal" subalgebra $D=\oplus_j\mathrm{Ext}^j_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})_{2j}\subset H_*(\Omega\mathrm{DJ};\mathbf{k})$ is isomorphic to the algebra
$$T(u_1,\dots,u_m)/(u_i^2=0;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K}).$$
3. If $\mathcal{K}$ is flag, then $H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})=D$.

It follows that $H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})$ and its subalgebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})$ admit a $\mathbb{Z}\times\mathbb{Z}^m_{\geq0}$-grading:

$$H_{-i,2\alpha}(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k}):=\mathrm{Ext}^i_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})_{2\alpha},\ \deg u_i=(-1,2e_i).$$

Generators and relations in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ (flag case)

Theorem [Vy22]. Let $\mathbf{k}$ be a field and $\mathcal{K}$ be a flag simplicial complex. Then

$$\boxed{\mathrm{Tor}_n^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})}(\mathbf{k},\mathbf{k})\cong\bigoplus_{J\subset[m]}\tilde{H}_{n-1}(\mathcal{K}_J;\mathbf{k}).}$$

Corollary. If $\mathcal{K}$ is flag and $\mathbf{k}$ is a field, any minimal presentation of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})$ consists of $\sum_{J\subset[m]}\tilde{b}_0(\mathcal{K}_J)$ generators and $\sum_{J\subset[m]}\tilde{b}_1(\mathcal{K}_J)$ relations (for each $J\subset[m]$, exactly $\tilde{b}_0(\mathcal{K}_J)$ generators and $\tilde{b}_1(\mathcal{K}_J)$ relations of degree $(-|J|,2\sum_{j\in J}e_j)\in\mathbb{Z}\times\mathbb{Z}^m_{\geq0}$).

An explicit minimal set of generators was constructed by Grbić, Panov, Theriault and Wu [GPTW16]: it consists of nested commutators

$$[\dots[[u_i,u_{j_1}],u_{j_2}],\dots,u_{j_\ell}],\ J=\{i\}\sqcup\{j_1<\dots<j_\ell\}$$

(for every $J\subset[m]$, choose $\tilde{b}_0(\mathcal{K}_J)$ elements i in different path components of $\mathcal{K}_J$ not containing $\max(J)$).

In upcoming work, we shall describe the relations. Recently Li Cai made some progress in this direction.

LS-category of $\mathcal{Z}_{\mathcal{K}}$ (flag case)

Theorem [Vy22]. For any flag simplicial complex $\mathcal{K}$,

$$\boxed{\mathrm{cat}\,\mathcal{Z}_{\mathcal{K}}=1+\max_{J\subset[m]}\mathrm{cdim}\,\mathcal{K}_J}$$

where $\mathrm{cdim}\,X:=\max\{k:\ \tilde{H}^k(X;\mathbb{Z})\neq0\}$.

Corollary [Vy22]. Let $\mathcal{K}$ be a flag triangulation of a d-manifold. Suppose that $\mathrm{sk}_i\mathcal{K}\subset\mathcal{L}\subset\mathrm{sk}_j\mathcal{K}$ for some $1\leq i\leq j\leq d$. Then

$$i+1\leq\mathrm{cat}(\mathcal{Z}_{\mathcal{L}})\leq j+1.$$

In particular, $\mathrm{cat}(\mathcal{Z}_{\mathcal{K}})=d+1$, $\mathrm{cat}(\mathcal{Z}_{\mathrm{sk}_i\mathcal{K}})=i+1$.

Upper bound is the combination of following results:

- $\mathrm{cat}\,\mathcal{Z}_{\mathcal{K}}\leq\mathrm{cat}(D^1,S^0)^{\mathcal{K}}$ by Beben and Grbić [BG21];
- $(D^1,S^0)^{\mathcal{K}}=K(\mathrm{RC}'_{\mathcal{K}},1)$ by Davis (here $\mathrm{RC}_{\mathcal{K}}$ is the right-angled Coxeter group);
- $\mathrm{cat}\,K(G,1)=\mathrm{cd}\,G$ by Eilenberg and Ganea;
- $\mathrm{cd}\,\mathrm{RC}'_{\mathcal{K}}=1+\max_{J\subset[m]}\mathrm{cdim}\,\mathcal{K}_J$ by Dranishnikov.

Further developments: nearly flag case

Definition. Simplicial complex $\mathcal{K}$ is nearly flag if any set of pairwise connected vertices spans a simplex or its boundary: $J\setminus\{i\}\in\mathcal{K},\ \forall i\in J$ whenever $\{i,j\}\in\mathcal{K},\ \forall i,j\in J$. For such J, we say that J is a hole if $J\notin\mathcal{K}$.

If $\mathcal{K}$ is nearly flag with holes $I_1,\dots,I_r$ then $\mathcal{K}^f=\mathcal{K}\cup\{I_1,\dots,I_r\}$ is a flag complex.

Theorem*. If $\mathcal{K}$ is nearly flag with holes $I_1,\dots,I_r$, then

$$H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbf{k})\cong T(u_1,\dots,u_m;w_1,\dots,w_r)/J,$$
$$J=(u_i^2=0,\ i=1,\dots,m;$$
$$u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K};$$
$$u_iw_s-w_su_i=0,\ s=1,\dots,r,\ i\in I_s),$$
$$\deg u_i=(-1,2e_i),\quad \deg w_s=(-2,2\sum_{i\in I_s}e_i).$$

Theorem*. If $\mathcal{K}$ is nearly flag with holes $I_1,\dots,I_r$, then $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbf{k})$ is isomorphic to the free product of the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}^f};\mathbf{k})$ and the tensor algebra on $\sum_{s=1}^r2^{m-|I_s|}$ generators

$$\kappa(J,s):=[\dots[[w_s,u_{j_1}],u_{j_2}],\dots u_{j_n}]$$

of degree $(-2-n,2\sum_{j\in J\sqcup I_s}e_j)$, $s=1,\dots,r$, $J=\{j_1<\dots<j_n\}\subset[m]\setminus I_s$.

The author would like to thank his advisor Taras E. Panov for his help, support, and valuable advice.

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This work was prepared within the framework of the HSE University Basic Research Program. The author is supported by the "BASIS" Foundation.